UNIT-I

Solution 1:- Given

$$A = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$$

We have to show that A is Hermitian and iA is skew-Hermitian.

$$\because \quad A = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$$

$$\Rightarrow \quad \bar{A} = \begin{bmatrix} 2 & 3-2i & -4 \\ 3+2i & 5 & -6i \\ -4 & 6i & 3 \end{bmatrix}$$

$$\Rightarrow \quad (\bar{A})^T = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix} = A$$

i.e. $A^\theta = A$, Thus A is Hermitian matrix.

$$\text{Now} \quad iA = i\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2i & 3i-2 & -4i \\ 3i+2 & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$$

$= B$ (say)

$$\because \quad B = \begin{bmatrix} 2i & 3i-2 & -4i \\ 3i+2 & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$$

$$\Rightarrow \quad \bar{B} = \begin{bmatrix} -2i & -3i-2 & 4i \\ -3i+2 & -5i & -6 \\ 4i & 6 & -3i \end{bmatrix}$$

$$\Rightarrow \quad (\bar{B})^T = \begin{bmatrix} -2i & -3i+2 & 4i \\ -3i-2 & -5i & 6 \\ 4i & -6 & -3i \end{bmatrix}$$

$$\Rightarrow \quad (\bar{B})^T = (-1)\begin{bmatrix} 2i & 3i-2 & -4i \\ 3i+2 & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$$

or $\quad B^\theta = (-1)B$

i.e. $\quad (iA)^\theta = (-1)(iA)$

Thus iA is skew-Hermitian matrix.

Q:2 (i) Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is Unitary.

(ii) If the Eigen value of the matrix A are 1, 1, 1 then find the Eigen value of $A^2+2A+3I$.

Solution 2:- (i) Let

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

We have to prove that A is unitary matr

$$\because \quad A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\Rightarrow \quad \bar{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$\Rightarrow \quad (\bar{A})^T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

or $$A^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

Now.

$$AA^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1\cdot1+(1+i)(1-i) & 1\cdot(1+i)+ \\ (1-i)\cdot1+(-1)(1-i) & (1-i)(1+i) \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1+2 & 0 \\ 0 & 2+1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

i.e. $AA^{\theta} = I$

Thus A is unitary matrix.

(ii) It is given that eigen values of the matrix A are 1, 1, 1. We have to find the eigen values of $A^2+2A+3I$

Since eigen values of the matrix A are 1, 1, 1, therefore, by the properties of eigen values,

eigen values of A^2 are $1^2, 1^2, 1^2$ i.e. 1, 1, 1.

eigen values of $2A$ are $2\cdot1, 2\cdot1, 2\cdot1$ i.e. 2, 2, 2

eigen values of $3I$ are 3, 3, 3.

$\therefore$ Eigen values of $A^2+2A+3I$ are $1+2+3$, $1+2+3$, $1+2+3$ i.e. 6, 6, 6.

Q:3 Compute the inverse of matrix $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ by employing elementary row transformations.

Solution 3:-

Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$.

We have to compute the inverse of the matrix A by employing elementry row transformations.

We have $A = I A$

i.e. $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 0 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \cdot A$$

Applying $R_2 \to -\frac{1}{2} \cdot R_2$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix} \cdot A$$

Applying $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1/2 & 0 \\ -2 & -1/2 & -1 \end{bmatrix} A$$

Applying $R_3 \to \left(-\frac{1}{7}\right) R_3$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1/2 & 0 \\ 2/7 & 1/14 & -1/7 \end{bmatrix}$$

Applying $R_1 \to R_1 - 3R_3$, $R_2 \to R_2 + 2R_3$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/7 & -3/14 & 3/7 \\ 11/7 & -5/14 & -2/7 \\ 2/7 & 1/14 & -1/7 \end{bmatrix}$$

Applying $R_1 \to R_1 - 3R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -32/7 & 12/14 & 9/7 \\ 11/7 & -5/14 & -2/7 \\ 2/7 & 1/14 & -1/7 \end{bmatrix}$$

$$I = \begin{bmatrix} -32/7 & 12/14 & 9/7 \\ 11/7 & -5/14 & -2/7 \\ 2/7 & 1/14 & -1/7 \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} -32/7 & 12/14 & 9/7 \\ 11/7 & -5/14 & -2/7 \\ 2/7 & 1/14 & -1/7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{[illegible]} \begin{bmatrix} -64 & 12 & 18 \\ 22 & -5 & -4 \\ [illegible] & 1 & -2 \end{bmatrix}$$

Q:4 Find the rank of the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

by reducing it to normal form.

Solution 4:-

Given-

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}_{4\times 4}$$

We have to find the rank of the matrix A by reducing it to normal form.

P.T.O

Applying $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Applying $C_2 \to C_2 + 1 \cdot C_1$, $C_3 \to C_3 + 2C_1$, $C_4 \to C_4 + 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & [illegible] & [illegible] & 10 \\ 6 & [illegible] & [illegible] & 17 \end{bmatrix}$$

Applying $R_2 \to R_2 + (-2)R_1$, $R_3 \to R_3 + (-3)R_1$, $R_4 \to R_4 + (-6)R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 3 & 12 & 17 \end{bmatrix}$$

Applying $R_2 \to R_2 + (-1)R_3$, $R_4 \to R_4 + (-2) \cdot R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

Applying $C_3 \to C_3 + 6C_2$, $C_4 \to C_4 + 3C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & [illegible] & 33 & 22 \\ 0 & [illegible] & 0 & 0 \end{bmatrix}$$

Applying $R_3 \to R_3 + (-4)R_2$, $R_4 \to R_4 + (-1)R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_3 \to \frac{1}{33} C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_4 \longrightarrow C_4 + (-22) C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} I_3 & 0_{3\times 1} \\ \hline 0_{1\times 3} & 0_{1\times 1} \end{array}\right]$$

Which is normal form.

$\therefore$ Rank (A) = 3

Q.5 Show that the system of equations $3x + 4y + 5z = A$, $4x + 5y + 6z = B$, $5x + 6y + 7z = C$ are consistent only if A, B & C are in arithmetic progression (A.P.).

Solution 5: Given system of equations is

$$3x + 4y + 5z = A,$$
$$4x + 5y + 6z = B,$$
$$5x + 6y + 7z = C.$$

We have to show that given system is CONSISTENT only if A, B and C are in A.P.

First of all we construct augmented matrix associated to this system.

$$\text{Augmented matrix} = \left[\begin{array}{ccc|c} 3 & 4 & 5 & A \\ 4 & 5 & 6 & B \\ 5 & 6 & 7 & C \end{array}\right]$$

[P : Q] (C say)

Applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & A \\ 1 & 1 & 1 & B-A \\ 2 & 2 & 2 & C-A \end{array}\right]$$

Applying $R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & B-A \\ 3 & 4 & 5 & A \\ 2 & 2 & 2 & C-A \end{array}\right]$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} & & 1 & B-A \\ & & 2 & 4A-3B \\ 0 & 0 & 0 & C-2B+A \end{array}\right]$$

$\therefore$ Which is echlon form.

For consistency,

$$\text{Rank}(P) = \text{Rank}([P \,\vdots\, Q])$$

i.e. we must have $C - 2B + A = 0$

$$\Rightarrow \quad 2B = A + C$$

$$\Rightarrow \quad B = \frac{1}{2}(A+C)$$

i.e. A, B, C are in arithmetic progression (A.P.).

Q:6 Test the consistency, if consistent Solve the system of equations :

$$10y + 3z = 0$$

$$3x + 3y + z = 1$$

$$2x - 3y - z = 5$$

and $$x + 2y = 4$$

Solution 6 :- Given system of equations is

$$\left.\begin{aligned} 10y + 3z &= 0 \\ 3x + 3y + z &= 1 \\ 2x - 3y - z &= 5 \\ x + 2y &= 4 \end{aligned}\right\} \quad \cdots\cdots (1)$$

Let the system (1) be denoted by $AX = B$. Then augmented matrix $[A \,\vdots\, B] = \left[\begin{array}{ccc|c} 0 & 10 & 3 & 0 \\ 3 & 3 & 1 & 1 \\ 2 & -3 & -1 & 5 \\ 1 & 2 & 0 & 4 \end{array}\right]$

Applying $R_1 \longleftrightarrow R_4$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 1 \\ 2 & -3 & -1 & 5 \\ 0 & 10 & 3 & 0 \end{array}\right]$$

Applying $R_2 \longrightarrow R_2 - 3R_1$, $R_3 \longrightarrow R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -11 \\ 0 & -7 & -1 & -3 \\ 0 & 10 & 3 & 0 \end{array}\right]$$

Applying $R_2 \longrightarrow \left(-\frac{1}{3}\right) R_1$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1/3 & 11/3 \\ 0 & -7 & -1 & -3 \\ 0 & 10 & 3 & 0 \end{array}\right]$$

Applying $R_3 \longrightarrow R_3 + 7R_2$, $R_4 \longrightarrow R_4 - 10R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1/3 & 11/3 \\ 0 & 0 & -10/3 & 68/3 \\ 0 & 0 & 13/3 & -110/3 \end{array}\right]$$

Applying $R_3 \longrightarrow \left(\frac{-3}{10}\right) R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1/3 & 11/3 \\ 0 & 0 & 1 & -68/10 \\ 0 & 0 & 19/3 & -110/3 \end{array}\right]$$

Applying $R_4 \longrightarrow R_4 - \frac{19}{3} R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1/3 & 11/3 \\ 0 & 0 & 1 & -68/10 \\ 0 & 0 & 0 & 64/10 \end{array}\right]$$

$\therefore$ Rank$(A) = 3$, Rank$([A:B]) = 4$

Since Rank$(A) \neq$ Rank$([A:B])$, therefore given system of equations is INCONSISTENT. This means that the system (1) has NO solution.

Q.7 For what values of λ & μ the system linear equations :

$$x + y + z = 6$$
$$x + 2y + 5z = 10 \quad \&$$
$$2x + 3y + \lambda z = \mu$$

has (i) a unique sol^n (ii) No sol^n & (iii) Infinite sol^n. Find the sol^n for $\lambda = 2$ & $\mu = 8$.

Solution 7:- Given system of linear equations is

$$x+y+z=6,$$
$$x+2y+5z=10 \quad \text{and}$$
$$2x+3y+\lambda z=\mu$$

Let the given system be denoted by $AX=B$. Then

augmented matrix $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 5 & 10 \\ 2 & 3 & \lambda & \mu \end{array}\right]$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 4 \\ 0 & 1 & \lambda-2 & \mu-12 \end{array}\right]$$

Applying $R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & \lambda-6 & \mu-16 \end{array}\right]$$

CASE 1:- If $\lambda \neq 6$, μ may have any value. Then

$\text{Rank}(A) = \text{Rank}([A:B]) = 3 =$ number of unknowns.

$\therefore$ The system has UNIQUE solution.

CASE 2:- If $\lambda = 6$, $\mu \neq 16$. Then

$\text{Rank}(A) = 2$, $\text{Rank}([A:B]) = 3$

$\because$ $\text{Rank}(A) \neq \text{Rank}([A:B])$

$\therefore$ The system has NO solution.

CASE 3:- If $\lambda = 6$, $\mu = 16$. Then

$\text{Rank}(A) = \text{Rank}([A:B]) = 2 <$ number of unknowns.

$\therefore$ The system has an INFINITE solutions.

Also, equivalent system of equations of given system is

$$\left.\begin{aligned} x+y+z&=6 \\ y+4z&=4 \\ (\lambda-6)z&=\mu-16. \end{aligned}\right\} \quad \cdots\cdots(1)$$

For $\lambda=2$ and $\mu=8$, system (1) becomes

$$\left.\begin{aligned} x+y+z&=6 \\ y+4z&=4 \\ -4z&=-8 \end{aligned}\right\} \quad \cdots\cdots(2)$$

From third equation of system (2), we have

$$z = 2.$$

Substitute this value in second equation of system (2), we get

$$y + 4(2) = 4$$

$$\Rightarrow \quad y = -4$$

Substitute $y = -4$ and $z = 2$ in first equation of system (2), we get

$$x + (-4) + 2 = 6$$

$$\Rightarrow \quad x = 8$$

Thus the solution ... $x = 8,\ y = -4,\ z = 2$ when $\lambda = 2,\ \mu = 8$

Q:8 (i) Define linearly dependent & linearly independent vectors.

(ii) If vectors $(0, 1, a)$, $(1, a, 1)$ & $(a, 1, 0)$ are linea[rly] dependent then find value of 'a'.

(iii) Show that $X_1 = (1, -1, 1)$, $X_2 = (2, 1, 1)$ & $X_3 = (3, 0$... are linearly dependent & find relation b/[w] them.

A(ii) Linearly Dependent Vectors (L.D.)

A set of r, n-tuple vectors $X_1, X_2, X_3 \dots X_r$ is said to be Linearly dependent if $\exists$ r-scalars $k_1, k_2, k_3 \dots k_r$ (Not all zero) such that

$$k_1X_1 + k_2X_2 + k_3X_3 + \dots k_rX_r = 0$$

Note → If a set of r-vectors are L.D then one member of set can be expressed as Linear combination of remaining vectors.

Linearly Independent Vectors. (L.I.)

A set of r, n-tuple vectors $X_1, X_2 \dots X_r$ is said to be Linearly Independent (L.I.) if all scalar are zero

$$k_1X_1 + k_2X_2 + k_3X_3 + \dots k_rX_r = 0 \Rightarrow k_1 = k_2 = k_3 = \dots = k_r = 0$$

Note Construct a matrix 'M' with the help of given vectors as columns.

if $\rho(M) =$ No of vectors $\Rightarrow$ L.I.

$\rho(M) <$ No of vectors $\Rightarrow$ L.D.

Q → B(ii) If vectors $(0,1,a)$; $(1,a,1)$ and $(a,1,0)$ are L.D., then find value of a

Sol

$$M = \begin{bmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix}.$$

Given vectors are L.D $\Rightarrow$ $\rho(M) < 3$ (No. of vectors)

$$\Rightarrow |M| = 0$$

$|M| = 0$

$\Rightarrow -1(-a) + a(1-a^2) = 0$

$\Rightarrow a + a - a^3 = 0 \Rightarrow (2a - a^3) = 0$

$a(2-a^2) = 0 \Rightarrow \boxed{a = 0, \pm\sqrt{2}}$

Q → B(iii) Show that $X_1 = (1,-1,1)$, $X_2 = (2,1,1)$, $X_3 = (3,0,2)$ are L.D. Find Relation b/w them.

Sol Let k_1, k_2, k_3 be scalars ——— ①

$k_1X_1 + k_2X_2 + k_3X_3 = 0$

Coeff Matrix $M = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 3 & 3 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

$\rho(M) = 2 <$ No of vectors (i.e 3) $\Rightarrow$ L.D.

Eqⁿ ① be relation b/w. vectors.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \Rightarrow \begin{matrix} k_1 + 2k_2 + 3k_3 = 0 \\ -k_2 - k_3 = 0 \end{matrix}$$

Let $k_3 = \lambda$

$k_2 = -\lambda$

$k_1 = -2k_2 - 3k_3 = -2(-\lambda) - 3\lambda = -\lambda$

so $\left.\begin{matrix} k_1 = -\lambda \\ k_2 = -\lambda \\ k_3 = \lambda \end{matrix}\right]$ Put these values in eqⁿ ①

$\Rightarrow -\lambda x_1 - \lambda x_2 + \lambda x_3 = 0$

$\Rightarrow \lambda x_3 = \lambda x_1 + \lambda x_2$

$\Rightarrow \boxed{x_3 = x_1 + x_2}$ Relation

Practice Questions

① Examine for linear dependence and find relation
$x_1 = (1,1,-1,1)$, $x_2 = (1,-1,-2,-1)$, $x_3 = (3,1,0,1)$

Ans:- L.D, $x_3 = 2x_1 + x_2$

(2) find whether or not set of vector are L.D or L.I.
$(1,1,0)$, $(1,0,1)$, $(0,1,1)$

Ans:- L.I.

(3) Show that $x_1 = (1,2,4)$, $x_2 = (2,-1,3)$, $x_3 = (0,1,2)$ and $x_4 = (-3,7,2)$ are L.D and find relation.

Ans:- L.D., $9x_1 = 12x_2 - 5x_3 + 5x_4$

Q.2 Find the Eigen values and the Corresponding Eigen vectors of the following matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

...tion 9 :- Given

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The characteristic equation of the given matrix A is

$$\det(A - \lambda I) = 0$$

or $$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(-2-\lambda)\{(1-\lambda)(0-\lambda) - (-2)(-6)\}$
$-2\{2(0-\lambda) - (-1)(-6)\} + (-3)\{2(-2)-(-1)(1-\lambda)\} = 0$

$\Rightarrow$ $(-2-\lambda)\{\lambda^2 - \lambda - 12\} - 2\{-2\lambda - 6\} - 3\{-3-\lambda\} = 0$

$\Rightarrow$ $(-2\lambda^2 + 2\lambda + 24) + (-\lambda^3 + \lambda^2 + 12\lambda)$
$+ (4\lambda + 12) + (3\lambda + 9) = 0$

$\Rightarrow$ $-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$

$\Rightarrow$ $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

By trial, $\lambda = -3$ satisfies it

$\therefore$ $(\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$

or $(\lambda+3)(\lambda+3)(\lambda-5) = 0$

$\Rightarrow$ $\lambda = -3, -3, 5$

Thus the eigen values of A are $-3, -3, 5$

- Corresponding to $\lambda = -3$, the eigen vectors are given by

$$(A + 3I)X_1 = 0$$

or $$\left\{\begin{bmatrix} -2 & 2 & -3 \\ 2 & [illegible] & [illegible] \\ -1 & [illegible] & [illegible] \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

or $$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get only one independent equation

$$x_1 + 2x_2 - 3x_3 = 0$$

Let $x_3 = k_1$, $x_2 = k_2$ (where either $k_1 \neq 0$ or $k_2 \neq 0$),

then $x_1 = 3k_1 - 2k_2$

$\therefore$ The eigen vectors are given by

$$X_1 = \begin{bmatrix} 3k_1 - 2k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

In particular, eigen vectors corresponding to $\lambda = -3$ are $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

- Now, corresponding to $\lambda = 5$, the eigen vectors are given by

$$(A - 5I) X_2 = 0$$

or

$$\left\{ \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left. \begin{aligned} -7y_1 + 2y_2 - 3y_3 &= 0 \\ 2y_1 - 4y_2 - 6y_3 &= 0 \\ -y_1 - 2y_2 - 5y_3 &= 0 \end{aligned} \right\}$$

From first two equations, we get

$$\frac{y_1}{2(-6) - (-4)(-3)} = \frac{-y_2}{(-3)(2) - (-6)(-7)} = \frac{y_3}{2(2)}$$

$$\Rightarrow \frac{y_1}{-24} = \frac{y_2}{-48} = \frac{y_3}{24}$$

$$\Rightarrow \frac{y_1}{1} = \frac{y_2}{2} = \frac{y_3}{-1} = k_3 \, (\neq 0) \quad \text{(say)}$$

$\therefore$ $y_1 = k_3$, $y_2 = 2k_3$, $y_3 = -k_3$

Hence the eigen vectors are given by $X_2 = k_3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Q.10 Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$ & verify Cayley Hamilton theorem. Hence

(i) Evaluate $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

(ii) Find A^{-1}.

Solution 10: Given

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

The characteristic equation of A is given by

$$\det(A - \lambda I) = 0$$

or $$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)\{(1-\lambda)(2-\lambda) - 1(0)\} - 1\{0(2-\lambda) - 1(0)\}$
$+ 1\{0.1 - 1.(1-\lambda)\} = 0$

$\Rightarrow (2-\lambda)(1-\lambda)(2-\lambda) - (1-\lambda) = 0$

$\Rightarrow (1-\lambda)\{4 - 4\lambda + \lambda^2\} - (1-\lambda) = 0$

$\Rightarrow (4 - 4\lambda + \lambda^2) - \lambda(4 - 4\lambda + \lambda^2) - (1-\lambda) = 0$

$\Rightarrow -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$. .... (1)

Which is required characteristic equation.

To verify Cayley-Hamilton theorem, we have to show that

$$A^3 - 5A^2 + 7A - 3I = 0 \qquad \text{----(2)}$$

Now,

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix},$$

$$A^3 = A^2 . A = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$\therefore \; A^3 - 5A^2 + 7A - 3I$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

(i) Now,

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$

$$= A^5\,0 + A\,0 + (A^2 + A + I)$$ [usi

$$= A^2 + A + I$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} +$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

(ii) From (2), we have

$$A^3 - 5A^2 + 7A - 3I = O$$

Pre-multiplying by A^{-1}, we get

$$A^2 - 5A + 7I - 3A^{-1} = O$$

$$\Rightarrow \quad 3A^{-1} = A^2 - 5A + 7I$$

$$\Rightarrow \quad A^{-1} = \frac{1}{3}(A^2 - 5A + 7I) \quad \text{....(3)}$$

Now $A^2 - 5A + 7I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$\therefore$ equation (3) gives

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

UNIT-2

UNIT - 02

1 if $y = \sin nx + \cos nx$, Prove that

$$y_r = n^r\left[1 + (-1)^r \sin 2nx\right]^{1/2}$$ where y_r is the

r^{th} differential coefficient of y w.r.to x.

Sol Given $y = \sin nx + \cos nx$

Diff r times w.r.to x both sides

$$y_r = n^r\left[\sin\left(nx + \frac{r\pi}{2}\right) + \cos\left(nx + \frac{r\pi}{2}\right)\right]$$

$$= n^r\left[\sin\left(nx + \frac{r\pi}{2}\right) + \cos\left(nx + \frac{r\pi}{2}\right)\right]^{2/2}$$

$$= n^r\left[\sin^2\left(nx + \frac{r\pi}{2}\right) + \cos^2\left(nx + \frac{r\pi}{2}\right) + 2\sin\left(nx + \frac{r\pi}{2}\right)\cdot\cos\left(nx + \frac{r\pi}{2}\right)\right]^{1/2}$$

$$= n^n\left[1 + \sin 2\left(nx + \frac{r\pi}{2}\right)\right]^{1/2}$$

$$= n^n\left[1 + \sin(2nx + n\pi)\right]^{1/2}$$

$$= n^n\left[1 + (-1)^r \sin 2nx\right]^{1/2}$$

$$\boxed{y_r = n^r\left[1 + (-1)^r \sin 2nx\right]^{1/2}}$$ $\left\{\because \sin(n\pi + \theta) = (-1)^n \sin\theta\right\}$

Hence proved.

Q-2:- if $y = \left(\frac{1+x}{1-x}\right)^{1/2}$, Prove that $(1-x^2)y_n - [2(n-1)x+1]y_{n-1} - (n-1)(\ldots$

Solution:- Given $y = \left(\frac{1+x}{1-x}\right)^{1/2}$

Taking log both sides

$$\log y = \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$$

Differentiating w.r.to x

$$\frac{1}{y}y_1 = \frac{1}{2}\left[\frac{1}{1+x} - \frac{(-1)}{1-x}\right]$$

$$\frac{y_1}{y} = \frac{1}{2}\left[\frac{1-x+1+x}{(1-x^2)}\right]$$

$$(1-x^2)\,y_1 = y \quad \text{— (1)}$$

Differentiating (1) w.r.to x $(n-1)$ times using Leibnitz T...

$${}^{n-1}C_0\,(y_1)_{n-1}\cdot(1-x^2) + {}^{n-1}C_1\,(y_1)_{n-2}(-2x) + {}^{n-1}C_2\,(y_1)_{n-3}(-2) = \ldots$$

$$\Rightarrow (1-x^2)\,y_n + (n-1)\,y_{n-1}(-2x) + \frac{(n-1)(n-2)}{\cancel{L2}}\,y_{n-2}(-\cancel{2}) - y_{\ldots}$$

$$\Rightarrow (1-x^2)\,y_n - 2(n-1)x\,y_{n-1} - (n-1)(n-2)\,y_{n-2} - y_{n-1} = \ldots$$

$$\Rightarrow \boxed{(1-x^2)y_n - [2(n-1)x+1]\,y_{n-1} - (n-1)(n-2)\,y_{n-2} = 0}$$

Hence Proved

Note:- ${}^{n-1}C_0 = 1$, ${}^{n-1}C_1 = (n-1)$, ${}^{n-1}C_2 = \frac{(n-1)(n\ldots}{L2}$

3. If $y = e^{\tan^{-1}x}$, Prove that $(1+x^2)y_{n+2} + [(2n+2)x - 1]y_{n+1} + n(n+1)y_n = 0$

Sol $y = e^{\tan^{-1}x}$

$y_1 = e^{\tan^{-1}x}\frac{1}{1+x^2} \Rightarrow (1+x^2)y_1 = e^{\tan^{-1}x} = y$

Again Diff, $(1+x^2)y_2 + 2xy_1 = y_1 \Rightarrow (1+x^2)y_2 + (2x-1)y_1 = 0$

Now Diff upto n times.

$\left[(1+x^2)y_{n+2} + ny_{n+1}(2x) + \frac{n(n-1)}{\cancel{L2}}y_n.\cancel{2}\right] + (2x-1)y_{n+1} + ny_n.2 = 0$

$\Rightarrow (1+x^2)y_{n+2} + y_{n+1}[(2n+2)x - 1] + (n^2-n+2n)y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + y_{n+1}[2(n+1)x-1] + (n^2+n)y_n = 0$ Proved

4. If $y^{1/m} + y^{-1/m} = 2x$ Prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Sol let $y^{1/m} = z$, $\frac{1}{z} = y^{-1/m}$

So $z + \frac{1}{z} = 2x \Rightarrow z^2 - 2zx + 1 = 0.$

$z = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$

$y^{1/m} = x + \sqrt{x^2-1}$ OR $y^{1/m} = x - \sqrt{x^2-1}$

$y = (x+\sqrt{x^2-1})^m$ OR $y = (x-\sqrt{x^2-1})^m$

$y_1 = m(x+\sqrt{x^2-1})^{m-1}\left[1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}\right]$

$= \frac{m(x+\sqrt{x^2-1})^m}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$

OR similarly $y = (x - \sqrt{x^2-1})^m$

$\Rightarrow (x^2-1)y_1^2 = m^2y^2$

$\Rightarrow (x^2-1)y_1^2 = m^2y^2$

$\Rightarrow (x^2-1)y_2(2y_1) + 2xy_1^2 = m^2 2yy_1$

$\Rightarrow (x^2-1)y_2 + xy_1 = m^2y$

Again Diff upto n times,

$\left[(x^2-1)y_{n+2} + ny_{n+1}(2x) + \frac{n(n-1)}{\cancel{2!}}y_n.\cancel{2}\right] + [xy_{n+1} + ny_n.1] = m^2y_n$

$\Rightarrow (x^2-1)y_{n+2} + xy_{n+1}(2n+1) + y_n(n^2-n+n-m^2) = 0$

$\Rightarrow (x^2-1)y_{n+2} + xy_{n+1}(2n+1) + (n^2-m^2)y_n = 0$ Proved

(5) If $\cos^{-1}x = (\log y)^{\frac{1}{m}}$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

Hence find $y_n(0)$.

Sol $\cos^{-1}x = \frac{1}{m}\log y \Rightarrow m\cos^{-1}x = \log y$ OR $y = e^{m\cos^{-1}x}$ — (1)

$\Rightarrow \frac{1}{y}y_1 = \frac{-m}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\,y_1 = -my$ | OR $y_1 = -e^{m\cos^{-1}x}\frac{m}{\sqrt{1-x^2}}$ — (2)

$\Rightarrow (1-x^2)y_1^2 = m^2y^2$

$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 = m^2 2y.y_1$

$\Rightarrow (1-x^2)y_2 - xy_1 = m^2y$ — (3)

Again Diff upto n times

$\Rightarrow \left[(1-x^2)y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2)\right] - [xy_{n+1} + ny_n] = m^2y_n$

$\Rightarrow (1-x^2)y_{n+2} - xy_{n+1}(2n+1) - y_n(n^2+m^2) = 0$ — (4)

To ~~fing~~ find $y_n(0)$, Put $x=0$ in eq^n (1), (2), (3), (4)

from Eq^n (1), $y(0) = e^{m\cos^{-1}0} = e^{m\pi/2}$

" (2), $y_1(0) = -me^{m\pi/2}$

" (3), $y_2(0) = m^2y(0) = m^2e^{m\pi/2}$

" (4), $\boxed{y_{n+2}(0) = (n^2+m^2)y_n(0)}$

Put $n=1$, $y_3(0) = (1^2+m^2)y_1(0) = -me^{m\pi/2}(1^2+m^2)$

$n=2$, $y_4(0) = (2^2+m^2)y_2(0) = m^2e^{m\pi/2}(2^2+m^2)$

$n=3$, $y_5(0) = (3^2+m^2)y_3(0) = -me^{m\pi/2}(1^2+m^2)(3^2+m^2)$

$$y_n(0) = \begin{cases} -me^{m\pi/2}(1^2+m^2)(3^2+m^2)\ldots[(n-2)^2+m^2] & n \text{ odd} \\ m^2e^{m\pi/2}(2^2+m^2)(4^2+m^2)\ldots[(n-2)^2+m^2] & n \text{ even} \end{cases}$$

6. $y = (\sin^{-1}x)^2$ then find $y_n(0)$

Sol $y = (\sin^{-1}x)^2$ — (1)

$y_1 = 2\sin^{-1}x \dfrac{1}{\sqrt{1-x^2}}$ — (2)

$\Rightarrow \sqrt{1-x^2}\, y_1 = 2\sin^{-1}x$

$\Rightarrow (1-x^2) y_1^2 = 4y$

$\Rightarrow (1-x^2) 2y_1y_2 - 2xy_1^2 = 4y_1$

$\Rightarrow (1-x^2) y_2 - xy_1 = 2$ — (3)

Again Diff upto n times.

$\left[(1-x^2)y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{2!} y_n(-2)\right] - \left[xy_{n+1} + ny_n\right] = 0$

$\Rightarrow (1-x^2)y_{n+2} - xy_{n+1}(2n+1) - n^2y_n = 0$ — (4)

Put $x=0$ in Eq^n (1),(2),(3),(4)

From Eq^n (1) , $y(0) = (\sin^{-1}0)^2 = 0$

" (2) , $y_1(0) = 0$

" (3) , $y_2(0) = 2$

" (4) , $\boxed{y_{n+2}(0) = n^2 y_n(0)}$

Put $n=1$ $y_3(0) = 1^2 y_1(0) = 0$

$n=2$ $y_4(0) = 2^2 y_2(0) = 2^2 . 2$

$n=3$, $y_5(0) = 3^2 y_3(0) = 0$

$n=4$, $y_6(0) = 4^2 y_4(0) = 4^2 . 2^2 . 2$

$\vdots$

$y_n(0) = \begin{cases} 0 & n \text{ odd} \\ 2 . 2^2 . 4^2 . 6^2 \ldots (n-2)^2, & n \neq 2,\ n \text{ even} \end{cases}$

7. $y = \sin(a\sin^{-1}x)$ find $y_n(0)$

Sol $y = \sin(a\sin^{-1}x)$ — (1)

$y_1 = \cos(a\sin^{-1}x) \cdot \dfrac{a}{\sqrt{1-x^2}}$ — (2)

$\Rightarrow \sqrt{1-x^2}\, y_1 = a\cos(a\sin^{-1}x)$

$\Rightarrow (1-x^2)y_1^2 = a^2\cos^2(a\sin^{-1}x) = a^2[1-\sin^2(a\sin^{-1}x)] = a^2[1-y^2]$

$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 = a^2[-2yy_1]$

$\Rightarrow (1-x^2)y_2 - xy_1 = -a^2y$ — (3)

Again diff upto n times,

$\Rightarrow \left[(1-x^2)y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{2!} y_n(-2)\right] - \left[xy_{n+1} + ny_n\right] = -a^2y_n$

$\Rightarrow (1-x^2)y_{n+2} - xy_{n+1}[2n+1] - y_n(n^2 - n + n - a^2) = 0$

$\Rightarrow (1-x^2)y_{n+2} - xy_{n+1}(2n+1) - (n^2-a^2)y_n = 0$ — (4)

Put $x=0$ in Eq^n (1),(2),(3),(4)

Eq^n (1) , $y(0) = \sin(a\sin^{-1}0) = 0$

(2) , $y_1(0) = a$

(3) , $y_2(0) = -a^2y(0) = 0$

(4) , $\boxed{y_{n+2}(0) = (n^2-a^2)\, y_n(0)}$

Put $n=1$ $y_3(0) = (1^2-a^2)\, y_1(0) = a(1^2-a^2)$

$n=2$ $y_4(0) = (2^2-a^2)\, y_2(0) = 0$

$n=3$ $y_5(0) = (3^2-a^2)\, y_3(0) = a(1^2-a^2)(3^2-a^2)$

$\vdots$

$y_n(0) = \begin{cases} 0 & \text{even} \\ a(1^2-a^2)(3^2-a^2)\ldots[(n-2)^2-a^2], & n \text{ odd} \end{cases}$

Q.8 (a) If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$.

(b) If $z = f(x+ct) + \phi(x-ct)$, show that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

Sol. (a) $\frac{\partial z}{\partial x} = \frac{1}{e^x+e^y}(e^x+0) = \frac{e^x}{e^x+e^y}$

$\frac{\partial z}{\partial y} = \frac{1}{e^x+e^y}(0+e^y) = \frac{e^y}{e^x+e^y}$

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{e^x(e^x+e^y) - e^x(e^x+0)}{(e^x+e^y)^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$

$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{0 - e^y(e^x+0)}{(e^x+e^y)^2} = \frac{-e^{x+y}}{(e^x+e^y)^2}$

or $s = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$

$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{e^y(e^x+e^y) - e^y(0+e^y)}{(e^x+e^y)^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$

Therefore,

$rt - s^2 = \frac{e^{x+y}}{(e^x+e^y)^2} \cdot \frac{e^{x+y}}{(e^x+e^y)^2} - \left[-\frac{e^{x+y}}{(e^x+e^y)^2}\right]^2$

$= \left[\frac{e^{x+y}}{(e^x+e^y)^2}\right]^2 - \left[\frac{e^{x+y}}{(e^x+e^y)^2}\right]^2 = 0$ Proved

Sol (b) We have $z = f(x+ct) + \phi(x-ct)$

$\frac{\partial z}{\partial x} = f'(x+ct).(1+0) + \phi'(x-ct)\ (1-0) = f'(x+ct) + \phi'(x-ct)$

$\frac{\partial^2 z}{\partial x^2} = f''(x+ct).1 + \phi''(x-ct).1$

$\frac{\partial z}{\partial t} = f'(x+ct)(0+c.1) + \phi'(x-ct)(0-c.1)$

$= c\,f'(x+ct) - c\,\phi'(x-ct)$

$\frac{\partial^2 z}{\partial t^2} = c\,f''(x+ct).c - c\,\phi''(x-ct).(-c)$

$= c^2[f''(x+ct) + \phi''(x-ct)] = c^2\frac{\partial^2 z}{\partial x^2}$ Proved

Q.9 (a) if $u = f(r)$, where $r^2 = x^2+y^2$, Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$.

Sol. When $r^2 = x^2+y^2$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$ and $2y\frac{\partial r}{\partial y} = 2y \Rightarrow \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$

Now $u = f(r) \Rightarrow \frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x}$

$\frac{\partial u}{\partial x} = f'(r).\frac{x}{r}$

Diff again w.r.to x

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(x.\frac{1}{r}.f'(r)\right) = 1.\frac{1}{r}f'(r) + x\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}.f'(r) + \frac{x}{r}f''(r)\frac{\partial r}{\partial x}$

$[D(uvw) = u_1vw + uv_1w + uvw_1]$

$= \frac{1}{r}f'(r) - \frac{x}{r^2}.\frac{x}{r}f'(r) + \frac{x}{r}.\frac{x}{r}f''(r)$

$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} - \frac{x^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r)$ —— (1)

Similarly $\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} - \frac{y^2}{r^3}f'(r) + \frac{y^2}{r^2}f''(r)$ —— (2)

Adding (1) and (2)

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2f'(r)}{r} - \frac{(x^2+y^2)}{r^3}f'(r) + \frac{(x^2+y^2)}{r^2}f''(r)$

$= \frac{2f'(r)}{r} - \frac{r^2}{r^3}f'(r) + \frac{r^2}{r^2}f''(r)$

$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)}$ Hence Proved

Q.9.(b). If $u = x\sin^{-1}\left(\frac{y}{x}\right) + y\sin^{-1}\left(\frac{x}{y}\right)$, evaluate

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$, (ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

(c). If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$, (ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u\sin u$

Sol. (a) $u(tx,ty) = tx\sin^{-1}\left(\frac{ty}{tx}\right) + ty\sin^{-1}\left(\frac{tx}{ty}\right)$

$= t\left[x\sin^{-1}\left(\frac{y}{x}\right) + y\sin^{-1}\left(\frac{x}{y}\right)\right] = t^1 u(x,y)$

∴ u is a homogeneous function in x and y of degree $n=1$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = u$

and $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 1(1-1)u = 0.$

(b) $u(tx,ty) = \tan^{-1}\frac{(tx)^3+(ty)^3}{tx-ty} = \tan^{-1} t^2\frac{x^3+y^3}{x-y} \neq t^n u(x,y)$

Here u is not a homogeneous function.

Let $\tan u = \frac{x^3+y^3}{x-y}$ is in the form of $F(u) = f(x,y)$

Now $f(tx,ty) = \frac{t^3(x^3+y^3)}{t(x-y)} = t^2 f(x,y)$

Hence $f(x,y) = F(u) = \tan u$ is homogeneous function of degree $n=2$

∴ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)} = \frac{2\tan u}{\sec^2 u} = 2\frac{\sin u}{\cos u}\cdot\cos^2 u$

$= 2\sin u\cos u = \sin 2u$

Also

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)[\phi'(u)-1]$, Here $\phi(u) = \sin 2u$ ∴ $\phi'(u) = 2\cos 2u$

$= \sin 2u[2\cos 2u - 1]$

$= 2\sin 2u\cos 2u - \sin 2u$

$= \sin 4u - \sin 2u = 2\cos 3u\sin u.$ Proved

Q10. (a). If $u = f(r,s,t)$ where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

(b) If $\phi(x,y,z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_z = -1$

Sol. (a) Given u is a function of r, s and t. Also r, s, t are functions of x, y and z. Therefore u is a composite function of x, y and z. Hence

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} = \frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0)$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$

L.H.S. $= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$= \left[\frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t}\right] + \left[-\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s}\right] + \left[-\frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}\right]$

$= \left(\frac{x}{y} - \frac{x}{y}\right)\frac{\partial u}{\partial r} + \left(\frac{y}{z} - \frac{y}{z}\right)\frac{\partial u}{\partial s} + \left(\frac{z}{x} - \frac{z}{x}\right)\frac{\partial u}{\partial t}$

$= 0 + 0 + 0 = 0 =$ R.H.S. Proved

(b) Taking y as a function of x and z. Treating x as constant $\left(\frac{\partial y}{\partial z}\right)_x = -\left[\frac{\partial\phi}{\partial z}\Big/\frac{\partial\phi}{\partial y}\right]$.

Treating z as a function of x and y, y as a constant

$\left(\frac{\partial z}{\partial x}\right)_y = -\left[\frac{\partial\phi}{\partial x}\Big/\frac{\partial\phi}{\partial z}\right]$.

Similarly $\left(\frac{\partial x}{\partial y}\right)_z = -\left[\frac{\partial\phi}{\partial y}\Big/\frac{\partial\phi}{\partial x}\right]$

L.H.S. $= \left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_z = \left[-\frac{\partial\phi/\partial z}{\partial\phi/\partial y}\right]\left[-\frac{\partial\phi/\partial x}{\partial\phi/\partial z}\right]\left[-\frac{\partial\phi/\partial y}{\partial\phi/\partial x}\right] =$ R

UNIT-3

Q1 Expand $\log(1+x)$ in powers of x. Then find series for $\log_e\left(\frac{1+x}{1-x}\right)$ and hence determine the value of $\log_e\left(\frac{11}{9}\right)$ upto five places of decimal.

Sol.

Let $f(x) = \log(1+x)$ $\quad \therefore f(0) = \log 1 = 0$

then $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$ $\quad \therefore f'(0) = 1$

$f''(x) = (-1)(1+x)^{-2}$ $\quad \therefore f''(0) = -1$

$f'''(x) = (-1)(-2)(1+x)^{-3}$ $\quad \therefore f'''(0) = 2$

$f^{iv}(x) = (-1)(-2)(-3)(1+x)^{-4}$ $\quad \therefore f^{iv}(0) = -6$

Putting these values in Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{iv}(0) + \cdots$$

$$\therefore \log(1+x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

or $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

changing x into $-x$, we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

Now

$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right]$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right]$$

Put $x = \frac{1}{10}$ in above

$$\log\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3}\left(\frac{1}{10}\right)^3 + \frac{1}{5}\left(\frac{1}{10}\right)^5 + \frac{1}{7}\left(\frac{1}{10}\right)^7 + \cdots\right]$$

$$= 0.20067.$$

Q2 (i) If $f(x) = x^3 + 8x^2 + 15x - 24$ Calculate $f\left(\frac{11}{10}\right)$ by using Taylor's series.

(ii) Expand $\sin x$ in ascending powers of $\left(x - \frac{\pi}{2}\right)$

Sol. By Taylor's Theorem,

(i) $f(x+h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \text{———}$

Put $x = 1$ and $h = \frac{1}{10}$

$$f\left(\frac{11}{10}\right) = f\left(1 + \frac{1}{10}\right) = f(1) + \frac{1}{10} f'(1) + \frac{1}{10^2} \cdot \frac{1}{\lfloor 2} f''(1) + \frac{1}{10^3} \frac{1}{\lfloor 3} f'''(1) + \text{———}$$

Now, $f(x) = x^3 + 8x^2 + 15x - 24$

$f'(x) = 3x^2 + 16x + 15$, $f''(x) = 6x + 16$, $f'''(x) = 6$

$\therefore f(1) = 0$, $f'(1) = 34$, $f''(1) = 22$, $f'''(1) = 6$

Hence $f\left(\frac{11}{10}\right) = 0 + \frac{1}{10} \cdot 34 + \frac{11}{100} + \frac{1}{1000} = 3.5111$ Ans

(ii) Another Form of Taylor's Series is

$$f(x) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \frac{h^3}{\lfloor 3} f'''(a) + \text{———}$$

Here $f(x) = \sin x$ and $a = \frac{\pi}{2}$ Put $h = x - a$, $h = x - \frac{\pi}{2}$

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$

$f\left(\frac{\pi}{2}\right) = 1$, $f'\left(\frac{\pi}{2}\right) = 0$, $f''\left(\frac{\pi}{2}\right) = -1$, $f'''\left(\frac{\pi}{2}\right) = 0$

Hence

$$f(x) = f\left(\frac{\pi}{2}\right) + \left(x - \frac{\pi}{2}\right) f'\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{\lfloor 2} f''\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^3}{\lfloor 3} f'''\left(\frac{\pi}{2}\right) + \text{———}$$

$\Rightarrow$

$$\sin x = 1 - \frac{1}{\lfloor 2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{\lfloor 4}\left(x - \frac{\pi}{2}\right)^4 \text{———}$$

Ans.

Q3. Expand $e^{ax} \sin by$ in powers of x and y as far the term of third degree.

Sol Maclaurin's Theorem is

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{\lfloor 2}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{\lfloor 3}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \text{———}$$

Now

$f(x,y) = e^{ax} \sin by$, $f(0,0) = 0$

$f_x(x,y) = a e^{ax} \sin by$, $f_x(0,0) = 0$

$f_y(x,y) = b e^{ax} \cos by$, $f_y(0,0) = b$

$f_{xx}(x,y) = a^2 e^{ax} \sin by$, $f_{xx}(0,0) = 0$

$f_{xy}(x,y) = ab e^{ax} \cos by$, $f_{xy}(0,0) = ab$

$f_{yy}(x,y) = -b^2 e^{ax} \sin by$, $f_{yy}(0,0) = 0$

$f_{xxx}(x,y) = a^3 e^{ax} \sin by$, $f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = a^2 b e^{ax} \cos by$, $f_{xxy}(0,0) = a^2 b$

$f_{xyy}(x,y) = -ab^2 e^{ax} \sin by$, $f_{xyy}(0,0) = 0$

$f_{yyy}(x,y) = -b^3 e^{ax} \cos by$, $f_{yyy}(0,0) = -b^3$

(Note- $e^0 = 1$, $\sin 0 = 0$, $\cos 0 = 1$)

Put all these values in eq①, we get

$$e^{ax} \sin by = 0 + [0 + y \cdot b] + \frac{1}{\lfloor 2}[0 + 2xy\, ab + 0] + \frac{1}{\lfloor 3}[3x^2y\, a^2b + y^3(-b^3)] + \text{———}$$

$$= by + abxy + \frac{1}{2} a^2 b x^2 y - \frac{1}{6} b^3 y^3 + \text{-----}$$

Ans

Q4. Expand $(x^2y + \sin y + e^x)$ in powers of $(x-1)$ and $(y-\pi)$.

Sol Taylor's Theorem is

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{\underline{2}}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots \quad \text{--- (1)}$$

Here $a=1$ and $b=\pi$.

Now

$f(x,y) = x^2y + \sin y + e^x$, $f(1,\pi) = \pi + e$

$f_x(x,y) = 2xy + e^x$, $f_x(1,\pi) = 2\pi + e$

$f_y(x,y) = x^2 + \cos y$, $f_y(1,\pi) = 1 - 1 = 0$

$f_{xx}(x,y) = 2y + e^x$, $f_{xx}(1,\pi) = 2\pi + e$

$f_{xy}(x,y) = 2x$, $f_{xy}(1,\pi) = 2$

$f_{yy}(x,y) = -\sin y$, $f_{yy}(1,\pi) = 0$

Put all these values in eq. (1),

$$f(x,y) = (\pi + e) + [(x-1)(2\pi + e) + 0] + \frac{1}{\underline{2}}[(x-1)^2(2\pi+e) + 2(x-1)(y-\pi)\cdot 2 + 0] + \cdots$$

$$= (\pi + e) + (x-1)(2\pi+e) + \frac{1}{2}(x-1)^2(2\pi+e) + 2(x-1)(y-\pi) + \cdots$$

Ans.

Question (5) Find the Extreme values of function $x^3 + y^3 - 3axy$

Solution (5) Here $f(x,y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$

$r = f_{xx} = 6x$, $s = f_{xy} = -3a$, $t = f_{yy} = 6y$

Put $f_x = 0$ and $f_y = 0$

$\Rightarrow x^2 - ay = 0$ --- (1) and $y^2 - ax = 0$ --- (2)

From (1) $y = \frac{x^2}{a}$, put this value of y in (2)

$\Rightarrow \frac{x^4}{a^2} - ax = 0$ OR $x(x^3 - a^3) = 0$

$\Rightarrow x = 0, a$

When $x = 0 \Rightarrow y = 0$, when $x = a \Rightarrow y = a$

$\therefore$ There are two stationary points $(0,0)$ and (a,a)

At $(0,0)$ $rt - s^2 = (36xy - 9a^2)_{(0,0)}$

$= -9a^2 < 0$ $\therefore$ No Extreme value at $(0,0)$

At (a,a) $rt - s^2 = 27a^2 > 0$

$\therefore f(x,y)$ has Extreme value at (a,a)

Now $r = 6a$

i) if $a > 0$, $r > 0$ $\therefore f(x,y)$ has a minimum value at (a,a)

Minimum value $= a^3 + a^3 - 3a^3 = \boxed{-a^3}$

ii) If $a < 0$, $r < 0$ $\therefore f(x,y)$ has a maximum value at (a,a)

Maximum value $= -a^3 - a^3 + 3a^3$

$= \boxed{a^3}$

Q6. Find the maximum and minimum distances of the point $(3,4,12)$ from the sphere $x^2+y^2+z^2=1$

Sol. Let (x,y,z) be any point on the sphere. Distance of the point $A(3,4,12)$ from (x,y,z) is given by $\sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$.

If the distance is maximum or minimum, so will be the square of the distance.

Let $f(x,y,z) = (x-3)^2+(y-4)^2+(z-12)^2$ — ①

subject to the condition that

$\phi(x,y,z) \equiv x^2+y^2+z^2-1=0$ — ②

Consider Lagrange's function

$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$

$= (x-3)^2+(y-4)^2+(z-12)^2+\lambda(x^2+y^2+z^2-1)$

For stationary values, $dF=0$

$\Rightarrow [2(x-3)+2\lambda x]dx + [2(y-4)+2\lambda y]dy + [2(z-12)+2\lambda z]dz = 0$

$\Rightarrow 2(x-3)+2\lambda x = 0$ — ③

$2(y-4)+2\lambda y=0$ — ④

$2(z-12)+2\lambda z=0$ — ⑤

Multiplying ③ by x, ④ by y, ⑤ by z and adding

$2(x^2+y^2+z^2)-6x-8y-24z+2\lambda(x^2+y^2+z^2)=0$

or $2-6x-8y-24z+2\lambda=0$

$3x+4y+12z=1+\lambda$ — ⑥

From ③, ④ and ⑤,

$x=\frac{3}{1+\lambda}$, $y=\frac{4}{1+\lambda}$, $z=\frac{12}{1+\lambda}$

Putting these values of x,y,z in ⑥, we have

$\frac{9}{1+\lambda}+\frac{16}{1+\lambda}+\frac{144}{1+\lambda}=1+\lambda$

$\Rightarrow (1+\lambda)^2=169$ or $1+\lambda=\pm 13$

$\therefore \lambda=12$ or $\lambda=-14$.

When $\lambda=12$, $x=\frac{3}{13}$, $y=\frac{4}{13}$, $z=\frac{12}{13}$

When $\lambda=-14$, $x=-\frac{3}{13}$, $y=-\frac{4}{13}$, $z=-\frac{12}{13}$

Hence $P\left(\frac{3}{13},\frac{4}{13},\frac{12}{13}\right)$ and $Q\left(-\frac{3}{13},-\frac{4}{13},-\frac{12}{13}\right)$

Now $PA=\sqrt{\left(3-\frac{3}{13}\right)^2+\left(4-\frac{4}{13}\right)^2+\left(12-\frac{12}{13}\right)^2}=12$

$QA=\sqrt{\left(3+\frac{3}{13}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2}=14$

Hence minimum distance $=12$ at point P and maximum distance $=14$ at point Q.

Q7. Find the dimensions of a rectangular box of maximum capacity whose surface area is given when

(a) box is open at the top

(b) box is closed

Sol. Let x,y,z be the length, breadth & height of the box, then volume $V=xyz$

and surface area $S=nxy+2yz+2zx$

$(n=1$, if box is open at the top and

$n=2$, if box is closed.

Consider Lagrange's function

$F(x,y,z) = xyz + \lambda(xxy + 2yz + 2zx - S)$

For stationary point $dF = 0$

$\Rightarrow [yz + \lambda(xy+2z)]\,dx + [xz + \lambda(xx+2z)]\,dy + [xy + \lambda(2y+2x)]\,dz = 0$

$\Rightarrow$ $yz + \lambda(xy+2z) = 0$ —— (1)

$xz + \lambda(xx+2z) = 0$ —— (2)

$xy + \lambda(2y+2x) = 0$ —— (3)

Multiplying Equation (1), (2), (3) by x, y, z respectively, we get

$xyz + \lambda(xxy + 2xz) = 0$ —— (4)

$xyz + \lambda(xxy + 2yz) = 0$ —— (5)

$xyz + \lambda(2yz + 2xz) = 0$ —— (6)

Equation (4) − Equation (5)

$\Rightarrow 2z\lambda(x-y) = 0 \Rightarrow \boxed{x = y}$

Equation (5) − Equation (6)

$\Rightarrow x\lambda(xy - 2z) = 0 \Rightarrow \boxed{xy = 2z}$

(a) When box is open at the top: put $x = 1$ $\therefore x = y$ and $y = 2z$

$\Rightarrow x = y = 2z$

$\therefore S = x^2 + x^2 + x^2 = 3x^2$

$\therefore$ dimensions are $x = \sqrt{\frac{S}{3}}$, $y = \sqrt{\frac{S}{3}}$

$z = \frac{1}{2}\sqrt{\frac{S}{3}}$

(b) When box is closed: put $x = 2$

$\therefore x = y$ and $y = z \Rightarrow x = y = z$

$\therefore S = 2x^2 + 2x^2 + 2x^2 = 6x^2$

OR $x = \sqrt{\frac{S}{6}}$, $y = \sqrt{\frac{S}{6}}$, $z = \sqrt{\frac{S}{6}}$

Question (8) If u, v, w are the roots of the Equation $(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$ then Find $\frac{\partial(u,v,w)}{\partial(a,b,c)}$

Solution (8) If u, v, w are the roots of the Equation $(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$

$(x^3 - a^3 + 3xa^2 - 3x^2a) + (x^3 - b^3 + 3xb^2 - 3x^2b) + (x^3 - c^3 + 3xc^2 - 3x^2c)$

$\Rightarrow 3x^3 - 3x^2(a+b+c) + 3x(a^2+b^2+c^2) - (a^3+b^3+c^3) = 0$

Then, $u + v + w = \frac{3(a+b+c)}{3}$

$uv + vw + wu = \frac{3(a^2+b^2+c^2)}{3}$

$uvw = \frac{a^3+b^3+c^3}{3}$

Let $F_1 \equiv u + v + w - a - b - c = 0$

$F_2 \equiv uv + vw + wu - a^2 - b^2 - c^2 = 0$

$F_3 \equiv uvw - \frac{1}{3}(a^3+b^3+c^3) = 0$

$$\frac{\partial(u,v,w)}{\partial(a,b,c)} = (-1)^3 \left[\frac{\partial(F_1,F_2,F_3)}{\partial(a,b,c)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} \right] \text{ —— (1)}$$

Now, $$\frac{\partial(F_1,F_2,F_3)}{\partial(a,b,c)} = \begin{vmatrix} -1 & -1 & -1 \\ -2a & -2b & -2c \\ -a^2 & -b^2 & -c^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$= \begin{vmatrix} -1 & 0 & 0 \\ -2a & -2(b-a) & -2(c-a) \\ -a^2 & -(b^2-a^2) & -(c^2-a^2) \end{vmatrix}$$

$$= (-1)(-2)(b-a)\{-(c-a)\} \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$$= -2(b-a)(c-a)(c+a-b-a)$$

$$= \boxed{-2(a-b)(b-c)(c-a)} \quad (2)$$

Also, $$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & uw & uv \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix}$$

$$= (u-v)(u-w) \begin{vmatrix} 1 & 0 & 0 \\ v+w & 1 & 1 \\ vw & w & v \end{vmatrix}$$

$$= (u-v)(v-w)(u-w)$$

$$= \boxed{-(u-v)(v-w)(w-u)} \quad (3)$$

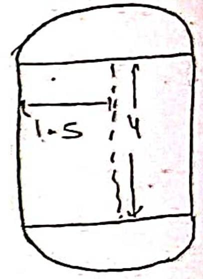
From equation (1), (2), (3)

$$\frac{\partial(u,v,w)}{\partial(a,b,c)} = -\left[\frac{2(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}\right]$$

Q9. A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m. and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05 m, find the percentage change in the volume of balloon.

Sol Here $r = 1.5$, $h = 4$

$\delta r = 0.01$, $\delta h = 0.05$



Let V be the volume of the balloon then

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3$$

$$\Rightarrow V = \pi r^2 h + \frac{4}{3}\pi r^3$$

Now $$\delta V = \pi \cdot 2r\,\delta r\, h + \pi r^2 \delta h + \frac{4}{3}\pi \cdot 3r^2 \delta r$$

$$\frac{\delta V}{V} = \frac{\pi r[2h\,\delta r + r\,\delta h + 4r\,\delta r]}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$= \frac{2(h+2r)\delta r + r\,\delta h}{rh + \frac{4}{3}r^2}$$

$$= \frac{2(4+3)(0.01) + (1.5)(0.05)}{(1.5)4 + \frac{4}{3}(1.5)^2}$$

$$= \frac{0.215}{9}$$

$$\therefore \frac{\delta V}{V} \times 100 = \frac{0.215}{9} \times 100 = 2.389\,\%$$

Question(10) Find approximate value of

$$\left[(0\cdot98)^2 + (2\cdot01)^2 + (1\cdot94)^2\right]^{1/2}$$

Solution(10) Let $f(x,y,z) = (x^2+y^2+z^2)^{1/2}$

Taking $x=1,\ y=2,\ z=2$ so that

$\delta x = -0\cdot02,\ \delta y = 0\cdot01,\ \delta z = -0\cdot06$

$$\frac{\partial f}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{1/2}},\quad \frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{1/2}}$$

$$\frac{\partial f}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{1/2}}.$$

Now, $df = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z$

$$\therefore\ df = \frac{1}{(x^2+y^2+z^2)^{1/2}}\left[x\delta x + y\delta y + z\delta z\right]$$

$$= \frac{1}{(1+4+4)^{1/2}}\left[-0\cdot02 + 0\cdot02 - 0\cdot12\right]$$

$$= \frac{1}{3}(-0\cdot12)$$

$$\boxed{df = -0\cdot04}$$

$$\therefore\ \left[(0\cdot98)^2 + (2\cdot01)^2 + (1\cdot94)^2\right]^{1/2} = f(1,2,2) + df$$

$$= (1+4+4)^{1/2} + (-0\cdot04)$$

$$= 3 - 0\cdot04$$

$$= \boxed{2\cdot96}$$

UNIT-4

UNIT – 4

Quest Evaluate the following integrals

(a) $\int_0^1 \int_0^{x^2} x e^y \, dy \, dx$

Ans Here $0 \le x \le 1$ and $0 \le y \le x^2$

So we have to integrate first with respect to y

$$\int_0^1 \int_0^{x^2} x e^y \, dy dx = \int_0^1 x \left[\int_0^{x^2} e^y \, dy \right] dx$$

$$= \int_0^1 x \left[e^y \right]_0^{x^2} dx$$

$$= \int_0^1 x \left[e^{x^2} - e^0 \right] dx$$

$$= \int_0^1 x (e^{x^2} - 1) \, dx$$

$$= \underbrace{\int_0^1 x e^{x^2} dx} - \int_0^1 x \, dx$$

Let $x^2 = u$

then $2x dx = du$

$\Rightarrow x dx = \frac{du}{2}$

$$= \int_0^1 \frac{e^u}{2} du - \int_0^1 x \, dx$$

$$= \frac{1}{2} (e^u)_0^1 - \left(\frac{x^2}{2} \right)_0^1$$

$$= \frac{1}{2}(e-1) - \frac{1}{2}$$

$$\boxed{\int_0^1 \int_0^{x^2} x e^y dy dx = \frac{1}{2}(e-2)}$$

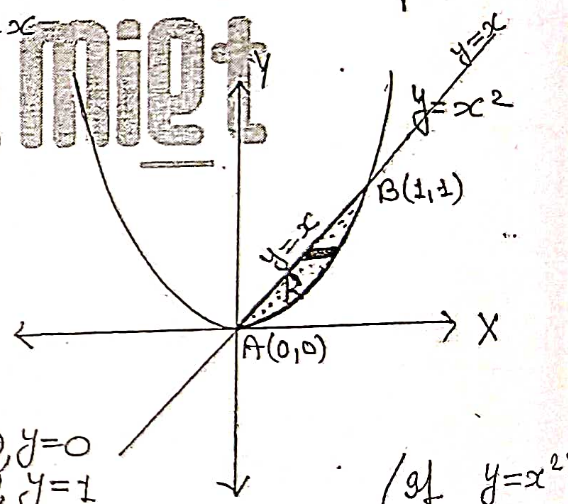
(b) $\iint_R xy(x+y) \, dx dy$ over the area between $y = x^2$ and $y = x$

Ans Given curves are parabola $y = x^2$ and straight line $y = x$

Now we find the intersection points of parabola $y = x^2$ and $y = x$

$y = x^2$ ——— (1)

$y = x$ ——— (2)



From (1) and (2)

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ or $x = 1$

If $x = 0$ then from (2), $y = 0$

If $x = 1$ then from (2), $y = 1$

So intersection points are given as

A(0,0) and B(1,1)

(If $y = x^2$
$\Rightarrow x = \sqrt{y}$)

Shaded portion R is the region of integration which is given by

$0 \le y \le 1$ and $y \le x \le \sqrt{y}$

Now $\iint_R xy(x+y)\,dx\,dy = \int_{y=0}^{1}\int_{x=y}^{\sqrt{y}} xy(x+y)\,dx\,dy$

$= \int_{y=0}^{1}\int_{x=y}^{\sqrt{y}} y(x^2+xy)\,dx\,dy$

$= \int_{y=0}^{1} y\left[\int_{y}^{\sqrt{y}}(x^2+xy)\,dx\right]dy$

$= \int_{y=0}^{1} y\left[\frac{x^3}{3}+\frac{x^2y}{2}\right]_y^{\sqrt{y}} dy$

$= \int_{y=0}^{1} y\left[\frac{y^{3/2}}{3}+\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^3}{2}\right]dy$

$= \int_{y=0}^{1}\left[\frac{y^{5/2}}{3}+\frac{y^3}{2}-\frac{y^4}{3}-\frac{y^4}{2}\right]dy$

$= \int_0^1 \frac{y^{5/2}}{3}+\frac{y^3}{2}-\frac{5y^4}{6}\Big]dy$

$= \left[\frac{1}{3}\cdot\frac{y^{(5/2+1)}}{(5/2)+1}+\frac{1}{2}\frac{y^4}{4}-\frac{5}{6}\frac{y^5}{5}\right]_0^1$

$= \left[\frac{1}{3}\frac{1}{(7/2)}+\frac{1}{2}\left(\frac{1}{4}\right)-\frac{1}{6}\right]$

$= \frac{2}{21}+\frac{1}{8}-\frac{1}{6} = \frac{16+21-28}{168}$

$= \frac{9}{168} = \frac{3}{56}$ Ans

Ques② (a) $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

Ans $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta$

$= \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a\sin\theta} d\theta$

$= \int_0^{\pi}\frac{1}{2}(a^2\sin^2\theta-0)\,d\theta$

$= \frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta$

$\left\{\begin{array}{l}\text{Since } \cos 2\theta = 1-2\sin^2\theta \\ \Rightarrow \sin^2\theta = \left(\frac{1-\cos 2\theta}{2}\right)\end{array}\right\}$

$= \frac{a^2}{2}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$

$= \frac{a^2}{4}\int_0^{\pi}(1-\cos 2\theta)\,d\theta$

$= \frac{a^2}{4}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi}$

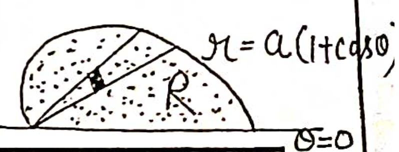
$= \frac{a^2}{4}[\pi-0-0+0]$

$= \left(\frac{\pi a^2}{4}\right)$ Ans

(b) Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the area of cardioid $r = a(1+\cos\theta)$ above the initial line

Ans Region R of integration is given in figure

Shaded region R represent it



Limits $0 \le \theta \le \pi$ and $0 \le r \le a(1+\cos\theta)$

Now $\iint_R r\sin\theta \, dr d\theta = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r\sin\theta \, dr d\theta$

$= \int_{\theta=0}^{\pi} \sin\theta \left[\int_{r=0}^{a(1+\cos\theta)} r \, dr \right] d\theta$

$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} d\theta$

$= \frac{1}{2} \int_0^{\pi} \sin\theta \left[a^2 (1+\cos\theta)^2 \right] d\theta$

$= \frac{a^2}{2} \int_0^{\pi} \sin\theta (1+\cos\theta)^2 d\theta$

Let $1+\cos\theta = u$
then $-\sin\theta \, d\theta = du$
$\Rightarrow \sin\theta \, d\theta = -du$
If $\theta = 0$ then $u = 2$
$\theta = \pi$ then $u = 0$

$= \frac{a^2}{2} \int_{u=2}^{0} u^2 (-du)$

$= -\frac{a^2}{2} \int_2^0 u^2 du$

$= -\frac{a^2}{2} \left[\frac{u^3}{3} \right]_2^0 = \left(\frac{-a^2}{6} \right)(0-8)$

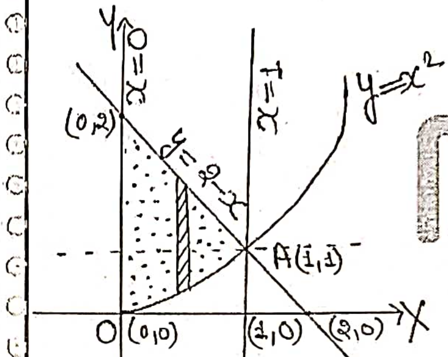
$$\boxed{\iint_R r\sin\theta \, dr d\theta = \frac{4a^2}{3}}$$

Ques③ Evaluate the integration by changing of order of integration $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

Ans Given $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

Here $0 \le x \le 1$ and $x^2 \le y \le 2-x$

so Region of integration is bounded by the curve $y = x^2$ (parabola), $y = 2-x$ (straight line), $x = 0$ (y-axis) and $x = 1$



Given Region

Here we have strip along Y-axis and limits are $x=0$ to $x=1$ and $y=x^2$ to $y=2-x$

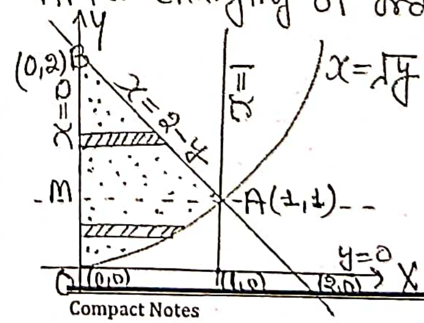
Intersection point of $y=x^2$ and $y=2-x^2$ is given as

$2-x^2 = x^2$
$\Rightarrow 2x^2 = 2$
$\Rightarrow x^2 = 1$
$\Rightarrow x = \pm 1$

If $x=1$ then $y=1$

$\boxed{\text{point } A(1,1)}$

After changing of order of integration—



After changing of integration we take strip (rectangular strip) along X-axis

To Evaluate the given integral after changing of integration region OABMO divided into two parts by AM as shown in figure—

So

$$I = \iint_{OAMO} xy\,dx\,dy + \iint_{MABM} xy\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} y\,dy + \int_1^2 y\left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\,dy$$

$$= \frac{1}{6}(1-0) + \frac{1}{2}\left(2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right)_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left[8 - \frac{32}{3} + 4\right] - \left[2 - \frac{4}{3} + \frac{1}{4}\right]\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(\frac{36-32}{3}\right) - \left(\frac{24-16+3}{12}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{4}{3} - \frac{11}{12}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{16-11}{12}\right]$$

$$= \frac{1}{6} + \frac{5}{24}$$

$$= \frac{4+5}{24}$$

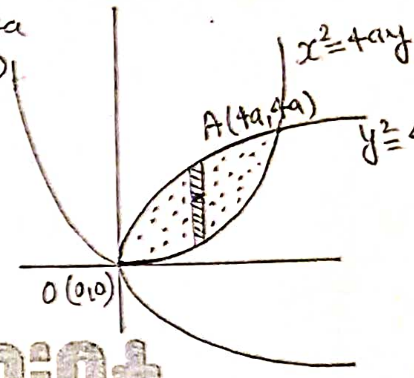
$$= \frac{9}{24}$$

$$\boxed{I = \frac{3}{8}}$$ Ans

Ques 4 (a) By Double integral show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$

Ans Given parabolas

$y^2 = 4ax$ —① and $x^2 = 4ay$ or $y = x^2/4a$ —②



From ① and ②

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

or $\frac{x^4}{16a^2} = 4ax$

or $x^4 = 64a^3x$

or $x(x^3 - 64a^3) = 0$

or $\boxed{x=0}$ or $x^3 = 64a^3$ $\Rightarrow \boxed{x = 4a}$

So if $x = 0 \Rightarrow y = 0$

if $x = 4a \Rightarrow y = 4a$

intersection points $O(0,0)$ and $A(4a,4a)$

To find the area of shaded region, take rectangular strip along y-axis.

so limits $\frac{x^2}{4a} \le y \le 2\sqrt{ax}$

and $0 \le x \le 4a$

$$\text{Area } A = \int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$$

$$= \int_0^{4a} \left[y\right]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx = \int_0^{4a}\left(2\sqrt{ax} - \frac{x^2}{4a}\right)dx$$

$$= 2\sqrt{a}\int_0^{4a}\sqrt{x}\,dx - \frac{1}{4a}\int_0^{4a} x^2\,dx$$

$$= 2\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right)_0^{4a} - \frac{1}{4a}\left(\frac{x^3}{3}\right)_0^{4a}$$

$$= 2\sqrt{a}\cdot\left(\frac{2}{3}\right)\left((4a)^{3/2}\right) - \frac{1}{12a}(4a)^3$$

$$= \left(\frac{4}{3}\sqrt{a}\right)\left(4^{3/2}a^{3/2}\right) - \frac{1}{12a}(64a^3)$$

$$= \left(\frac{4}{3}\sqrt{a}\right)\left(8a^{3/2}\right) - \frac{64}{12}a^2$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

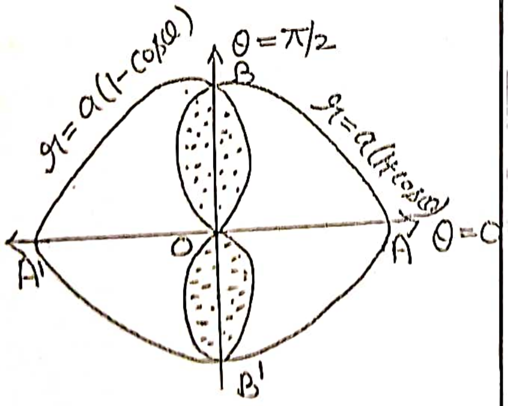
$$\boxed{\text{Area} = \frac{16}{3}a^2}$$ Ans

Ques (4)(b) Show that the area common to the Cardioids $r = a(1-\cos\theta)$ and $r = a(1+\cos\theta)$ is $\left(\frac{3\pi}{2} - 4\right)a^2$.

Ans Given Cardioids are

$r = a(1+\cos\theta)$ —— (1)

$r = a(1-\cos\theta)$ —— (2)



From (1) and (2)

$a(1+\cos\theta) = a(1-\cos\theta)$

or $1+\cos\theta = 1-\cos\theta$

or $2\cos\theta = 0$

or $\cos\theta = 0$

or $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

Common Area to the cardiode

$= 4$ (Common area in first quadrant)

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta$$

$$= 2\int_0^{\pi/2} a^2(1-\cos\theta)^2\,d\theta$$

$$= 2a^2\int_0^{\pi/2}(1+\cos^2\theta - 2\cos\theta)\,d\theta$$

$$= 2a^2\int_0^{\pi/2}\left(1+\left(\frac{1+\cos 2\theta}{2}\right) - 2\cos\theta\right)d\theta$$

$$= 2a^2\int_0^{\pi/2}\left\{1+\frac{1}{2}+\frac{\cos 2\theta}{2} - 2\cos\theta\right\}d\theta$$

$$= 2a^2\int_0^{\pi/2}\left\{\frac{3}{2}+\frac{\cos 2\theta}{2} - 2\cos\theta\right\}d\theta$$

$$= 2a^2\left[\frac{3}{2}\theta + \frac{\sin 2\theta}{4} - 2\sin\theta\right]_0^{\pi/2}$$

$$= 2a^2\left[\frac{3\pi}{4} + 0 - 2\right]$$

$$= 2a^2\left[\frac{3\pi}{4} - 2\right]$$

$$\boxed{\text{Common Area} = \frac{a^2}{2}(3\pi - 8)}$$ Ans.

or

$$\boxed{\text{Common Area} = a^2\left(\frac{3\pi}{2} - 4\right)}$$

Ques 5) Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$

Sol^n $I = \int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$

$= \int_0^1\int_0^1\int_0^1 e^x.e^y.e^z\,dx\,dy\,dz$

$= \int_0^1\int_0^1 e^y.e^z\,(e^x)_0^1\,dy\,dz$

$= \int_0^1\int_0^1 e^y e^z (e-1)\,dy\,dz$

$= (e-1)\int_0^1 e^z\,(e^y)_0^1\,dz$

$= (e-1)\int_0^1 e^z\,(e-1)\,dz$

$= (e-1)^2\int_0^1 e^z\,dz$

$= (e-1)^2\,(e^z)_0^1$

$= (e-1)^2\,(e-1)$

$= (e-1)^3$ An.

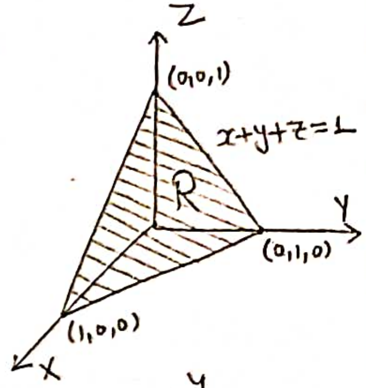
Ques (5) Calculate the volume of the solid bounded by $x=0,\ y=0,\ z=0$ and $x+y+z=1$.

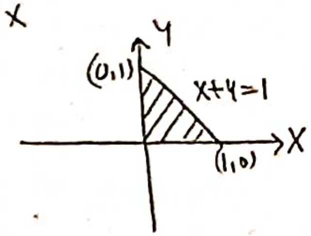
Ans Now limits for the integration

$0 \le z \le 1-x-y$

$0 \le y \le 1-x$

$0 \le x \le 1$





In X-Y plane

Volume $V = \iiint_R dz\,dy\,dx$

$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dz\,dy\,dx$

$= \int_0^1\int_0^{1-x} [z]_0^{1-x-y}\,dy\,dx$

$= \int_0^1\int_0^{1-x} (1-x-y)\,dy\,dx$

$= \int_0^1\int_0^{1-x} (1-x)\,dy\,dx - \int_0^1\int_0^{1-x} y\,dy\,dx$

$= \int_0^1 (1-x)\,[y]_0^{1-x}\,dx - \int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$

$= \int_0^1 (1-x)^2\,dx - \frac{1}{2}\int_0^1 (1-x)^2\,dx$

$= \frac{1}{2}\int_0^1 (1-x)^2\,dx$

$= \frac{1}{2}\left[-\frac{(1-x)^3}{3}\right]_0^1$

$= \frac{1}{6}\left[-(1-1)^3 + (1-0)^3\right]$

$= \frac{1}{6}\,[0+1]$

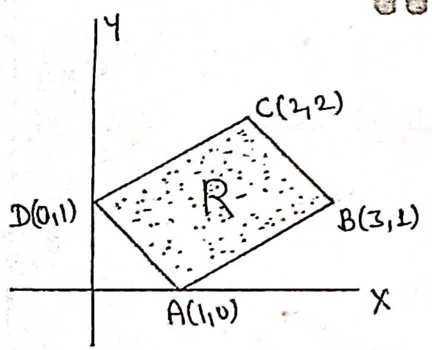
$\boxed{\text{Volume} = 1/6}$

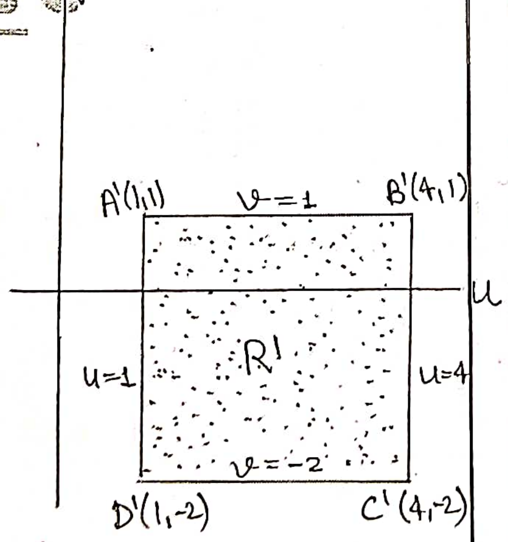
Ques 7 (a) Evaluate $\iint_R (x+y)^2 dxdy$, where R is the Parallelogram in the xy-plane with vertices (1,0), (3,1), (2,2), (0,1). using the transformation $u = x+y$, $v = x-2y$

Ans The vertices A(1,0), B(3,1), C(2,2), D(0,1) of the parallelogram ABCD in xy-plane become A'(1,1), B'(4,1), C'(4,-2) and D'(1,-2) in the uv-plane by using the transformation. $u = x+y$ & $v = x-2y$

The region R in xy-plane becomes the region R' in the uv-plane which is a rectangle bounded by the line $u=1$, $u=4$, $v=-2$, $v=1$. Solving the given equations ~~for x and y~~ we get

$$x = \frac{1}{3}(2u+v), \quad y = \frac{1}{3}(u-v)$$





$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{vmatrix} = -1/3$$

$$dxdy = |J|\,dudv = \left|\frac{-1}{3}\right| dudv = \frac{1}{3} dudv$$

$$\iint_R (x+y)^2 dxdy = \iint_{R'} u^2 |J|\, dudv = \iint_{R'} u^2 \left(\frac{1}{3}\right) dudv$$

$$= \frac{1}{3}\int_{-2}^{1}\int_{1}^{4} u^2\, dudv$$

$$= \frac{1}{3}\int_{-2}^{1} \left(\frac{u^3}{3}\right)_1^4 dv$$

$$= \frac{1}{3}\cdot\frac{1}{3}\int_{-2}^{1} (64-1)\, dv$$

$$= \frac{63}{9}\left[v\right]_{-2}^{1}$$

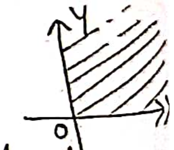
$$= 7\times 3$$

$$= 21 \quad \text{Ans}$$

Ques 7 (b) Change into polar coordinates & evaluate

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\, dx$$

Ans Given Limits $0 \le x < \infty$ and $0 \le y < \infty$

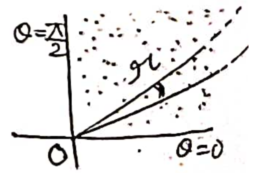


Given region of integration is in Ist quadrant.

To change it into polar Coordinate we have $x = r\cos\theta$, $y = r\sin\theta$

$$x^2 + y^2 = r^2$$

So $0 \le r < \infty$ and $0 \le \theta \le \frac{\pi}{2}$



Also $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$J = r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta)$

$\boxed{J = r}$

so $dy\,dx = |J|\,dr\,d\theta$

$dy\,dx = r\,dr\,d\theta$

so $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dy\,dx = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$

$\left\{\begin{array}{l} \text{let } r^2 = u \\ \Rightarrow 2r\,dr = du \\ r\,dr = \frac{du}{2} \\ r = 0 \Rightarrow u = 0 \\ r = \infty \Rightarrow u = \infty \end{array}\right\}$

$= \int_{\theta=0}^{\pi/2}\int_{u=0}^{\infty} \frac{1}{2}e^{-u}du\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\left[-e^{-u}\right]_0^\infty d\theta$

$= \frac{1}{2}\int_0^{\pi/2}(-e^{-\infty}+e^0)\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}(-0+1)\,d\theta$

$= \frac{1}{2}\left[\theta\right]_0^{\pi/2}$

$= \frac{1}{2}\cdot\frac{\pi}{2}$

$\boxed{\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dy\,dx = \frac{\pi}{4} \text{ Ans}}$

(3) (a) Use Beta function to evaluate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$

Ans (formula used) 1) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}}$

$\int_0^\infty \frac{x^8(1-x^6)\,dx}{(1+x)^{24}} = \int_0^\infty \frac{x^8\,dx}{(1+x)^{24}} - \int_0^\infty \frac{x^{14}\,dx}{(1+x)^{24}}$

$= \int_0^\infty \frac{x^{9-1}\,dx}{(1+x)^{9+15}} - \int_0^\infty \frac{x^{15-1}\,dx}{(1+x)^{15+9}}$

$= \beta(9,15) - \beta(15,9) = 0$

$(\because \beta(m,n) = \beta(n,m))$

(b) find the value of $\Gamma\left(-5/2\right)$

formula used

$\Gamma(n+1) = n\Gamma(n)$

$\Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n}$

Ans $\Gamma\left(\frac{-5}{2}\right) = \frac{\Gamma\left(\frac{-5}{2}+1\right)}{\left(\frac{-5}{2}\right)}$

$= \frac{\Gamma(-3/2)}{(-5/2)} = \frac{\Gamma\left(\frac{-3}{2}+1\right)}{\left(\frac{-5}{2}\right)\left(\frac{-3}{2}\right)}$

$= \frac{\Gamma(-1/2)}{\left(\frac{-5}{2}\right)\left(\frac{-3}{2}\right)} = \frac{\Gamma\left(\frac{-1}{2}+1\right)}{\left(\frac{-5}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-1}{2}\right)}$

$= \frac{\Gamma(1/2)}{\left(\frac{-5}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-1}{2}\right)} = \frac{\sqrt{\pi}}{(-15/8)}$

$\boxed{\Gamma\left(\frac{-5}{2}\right) = -\frac{8}{15}\sqrt{\pi}}$

(c) Define Gamma & Beta function. Prove that

$$\beta(l,m)\beta(l+m,n)\beta(l+m+n,p) = \frac{\Gamma l\,\Gamma m\,\Gamma n\,\Gamma p}{\Gamma(l+m+n+p)}$$

Ans <u>Gamma function</u> : Gamma function is denoted as Γn, $n>0$ and defined as

$$\boxed{\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx}$$

<u>Beta function</u> : Beta function is denoted as $\beta(m,n)$; $m>0$, $n>0$ and defined as.

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx}$$

Formula used $\rightarrow \beta(a,b) = \frac{\Gamma a\,\Gamma b}{\Gamma(a+b)}$.

L.H.S $\beta(l,m)\,\beta(l+m,n)\,\beta(l+m+n,p)$

$$= \left(\frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}\right)\left(\frac{\Gamma(l+m)\,\Gamma n}{\Gamma(l+m+n)}\right)\left(\frac{\Gamma(l+m+n)\,\Gamma p}{\Gamma(l+m+n+p)}\right)$$

$$= \frac{\Gamma l\,\Gamma m\,\Gamma n\,\Gamma p}{\Gamma(l+m+n+p)} = \text{R.H.S}$$

9(a) Apply Dirichlet integral to find the volume of an octant of the sphere $x^2+y^2+z^2=25$

Ans Sphere $x^2+y^2+z^2=25$

also given an octant let us we take Ist octant.

$$\left(\frac{x}{5}\right)^2+\left(\frac{y}{5}\right)^2+\left(\frac{z}{5}\right)^2=1$$

let $\left(\frac{x}{5}\right)^2 = u$, $\left(\frac{y}{5}\right)^2 = v$, $\left(\frac{z}{5}\right)^2 = w$

$x = 5\sqrt{u}$, $y = 5\sqrt{v}$, $z = 5\sqrt{w}$

$dx = \frac{5\,du}{2\sqrt{u}}$, $dy = \frac{5\,dv}{2\sqrt{v}}$, $dz = \frac{5\,dw}{2\sqrt{w}}$

also $u+v+w=1$

Required Volume $= \iiint dx\,dy\,dz$

$$= \iiint \left(\frac{5\,du}{2\sqrt{u}}\right)\left(\frac{5\,dv}{2\sqrt{v}}\right)\left(\frac{5\,dw}{2\sqrt{w}}\right)$$

$$= \frac{125}{8}\iiint u^{-1/2}v^{-1/2}w^{-1/2}\,du\,dv\,dw$$

$$= \frac{125}{8}\iiint u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\,du\,dv\,dw$$

$$= \frac{125}{8}\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\right)} = \frac{125}{8}\,\frac{\left(\Gamma\frac{1}{2}\right)^3}{\Gamma\left(\frac{3}{2}+1\right)}$$

$$= \frac{125}{8}\,\frac{\left(\Gamma\frac{1}{2}\right)^3}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}} = \frac{125\times 4}{8\times 3}\left(\Gamma\tfrac{1}{2}\right)^2$$

$$= \frac{125}{6}(\sqrt{\pi})^2 \qquad \left(\because \Gamma\tfrac{1}{2} = \sqrt{\pi}\right)$$

$$= \left(\frac{125\pi}{6}\right)$$

<u>Dirchlet formula</u> · $\iiint_V x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$

Where $x+y+z=1$

(3) Find the volume and the mass contained in the solid region in the first octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ if the density at any point $f(x,y,z)=kxyz$

Sol. Put $\frac{x^2}{a^2}=u,\ \frac{y^2}{b^2}=v,\ \frac{z^2}{c^2}=w$

$$x=a\sqrt{u},\quad y=b\sqrt{v},\quad z=c\sqrt{w}$$

$$dx=\frac{a\,du}{2\sqrt{u}},\quad dy=\frac{b\,dv}{2\sqrt{v}},\quad dz=\frac{c\,dw}{2\sqrt{w}}$$

then $u\geq0,\ v\geq0,\ w\geq0$ and $u+v+w=1$

Required Volume $=\iiint dx\,dy\,dz$

$$=\iiint \frac{abc\,du\,dv\,dw}{8\sqrt{u}\sqrt{v}\sqrt{w}}$$

$$=\frac{abc}{8}\iiint u^{-1/2}v^{-1/2}w^{-1/2}\,du\,dv\,dw$$

$$=\frac{abc}{8}\iiint u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\,du\,dv\,dw$$

$$=\frac{abc}{8}\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\right)}=\frac{abc}{8}\frac{\left(\Gamma\frac{1}{2}\right)^3}{\Gamma\left(\frac{3}{2}+1\right)}$$

$$=\frac{abc}{8}\frac{\left(\Gamma\frac{1}{2}\right)^3}{\frac{3}{2}\cdot\frac{1}{2}\left(\Gamma\frac{1}{2}\right)}=\frac{abc}{6}\left(\Gamma\frac{1}{2}\right)^2$$

$$\boxed{\text{Required Vol.}=\frac{\pi abc}{6}}$$

Required Mass $=\iiint kxyz\,dx\,dy\,dz$

$$=\iiint k(a\sqrt{u})(b\sqrt{v})(c\sqrt{w})\frac{abc\,du\,dv\,dw}{8\sqrt{u}\sqrt{v}\sqrt{w}}$$

$$=k\frac{a^2b^2c^2}{8}\iiint du\,dv\,dw$$

$$=\frac{ka^2b^2c^2}{8}\iiint u^{1-1}v^{1-1}w^{1-1}\,du\,dv\,dw$$

$$=\frac{ka^2b^2c^2}{8}\frac{\Gamma1\Gamma1\Gamma1}{\Gamma(1+1+1+1)}$$

$$=\frac{ka^2b^2c^2}{8}\cdot\frac{1}{\Gamma4}$$

$$=\frac{ka^2b^2c^2}{8}\cdot\frac{1}{6}$$

$$=\frac{ka^2b^2c^2}{48}$$

$\boxed{\text{Formula Used}}$

$$\iiint x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}\quad ;\ x+y+z\leq1$$

10.) Show that $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}$ the integral being extended to all positive values of the variables for which the expression is real.

Ans Formula Used

$$\iiint f(x+y+z)\,x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n)}\int_{h_1}^{h_2} f(u)\,u^{l+m+n-1}\,du$$

where $h_1 < x+y+z < h_2$

→ The expression will be real, if

$$1-x^2-y^2-z^2 > 0$$
$$x^2+y^2+z^2 < 1$$

Hence the given integral is extended for all positive value of x, y, z such that

$$0 < x^2+y^2+z^2 < 1$$

Let $x^2 = u$, $\quad y^2 = v$, $\quad z^2 = w$

$x = \sqrt{u}$, $\quad y = \sqrt{v}$, $\quad z = \sqrt{w}$

$dx = \frac{du}{2\sqrt{u}}$, $\quad dy = \frac{dv}{2\sqrt{v}}$, $\quad dz = \frac{dw}{2\sqrt{w}}$

then the condition becomes.

$$0 < u+v+w < 1$$

Now integral becomes.

$$\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \iiint \frac{\left(\frac{du}{2\sqrt{u}}\right)\left(\frac{dv}{2\sqrt{v}}\right)\left(\frac{dw}{2\sqrt{w}}\right)}{\sqrt{1-(u+v+w)}}$$

$$= \frac{1}{8}\iiint \frac{u^{-1/2}v^{-1/2}w^{-1/2}}{\sqrt{1-(u+v+w)}}\,du\,dv\,dw$$

$$= \frac{1}{8}\iiint \frac{u^{1/2-1}v^{1/2-1}w^{1/2-1}}{\sqrt{1-(u+v+w)}}\,du\,dv\,dw$$

$$= \frac{1}{8}\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)}\int_0^1 \frac{1}{\sqrt{1-t}}\,t^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\,dt$$

$$= \frac{1}{8}\frac{\left(\Gamma\frac{1}{2}\right)^3}{\left(\Gamma\frac{3}{2}\right)}\int_0^1 \frac{t^{1/2}}{\sqrt{1-t}}\,dt$$

$$= \frac{1}{8}\cdot\frac{\left(\Gamma\frac{1}{2}\right)^3}{\left(\frac{1}{2}\right)\Gamma\frac{1}{2}}\int_0^1 \frac{t^{1/2}\,dt}{\sqrt{1-t}}$$

$$= \frac{\pi}{4}\int_0^1 \frac{t^{1/2}\,dt}{\sqrt{1-t}} = \frac{\pi}{4}\int_0^1 \frac{t^{1/2}}{\sqrt{1-t}}\,dt$$

Let $t = \sin^2\theta \Rightarrow dt = 2\sin\theta\cos\theta\,d\theta$

$$= \frac{\pi}{4}\int_0^{\pi/2}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\cdot(2\sin\theta\cos\theta)\,d\theta$$

$$= \frac{\pi}{4}\int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}(2\sin\theta\cos\theta)\,d\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/2}\sin^2\theta\cos^0\theta\,d\theta$$

$$= \frac{\pi}{2}\,\frac{\Gamma\frac{2+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{2+0+2}{2}} = \frac{\frac{\pi}{2}\Gamma\frac{3}{2}\Gamma\frac{1}{2}}{2\Gamma 2}$$

$$= \frac{\frac{\pi}{2}\cdot\frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2}{2} = \frac{\pi}{8}\left(\Gamma\frac{1}{2}\right)^2$$

$$= \frac{\pi}{8}\left(\sqrt{\pi}\right)^2$$

$$= \frac{\pi^2}{8} = \text{R.H.S}$$

UNIT-5

Unit V : Vector Calculus

Q.1 (i) If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$ prove that grad u, grad v and grad w are coplanar vectors.
(ii) Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$.

Soln (i) $\text{grad } u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x+y+z) = \hat{i}+\hat{j}+\hat{k}$

$$\text{grad } v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad } w = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial y}\right)(yz+zx+xy) = \hat{i}(x+y) + \hat{j}(z+x) + \hat{k}(y+x)$$

For coplanarity $[\vec{a}\cdot(\vec{b}\times\vec{c})] = 0$ triple product

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\boxed{\text{grad } u\cdot(\text{grad } v \times \text{grad } w)} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x+y & z+x & y+x \end{vmatrix}$$

$$\Rightarrow 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y & z+x & y+z \end{vmatrix} \Rightarrow 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ x+y & z+x & y+z \end{vmatrix}$$

Applying $R_2 \to R_2 + R_3$

$$\Rightarrow 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ x+y & z+x & y+z \end{vmatrix} = \boxed{0}$$

Hence grad u, grad v and grad w are coplanar vectors

(ii) Angle between two surfaces at a point is the angle between the normals to the surfaces at that point

Let $\phi_1 = x^2+y^2+z^2-9 = 0$ and $\phi_2 = x^2+y^2-3-z = 0$

$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ and $\nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$\nabla\phi_1|_{(2,-1,2)} = \vec{n_1} = 4\hat{i} - 2\hat{j} + 4\hat{k}$, $\nabla\phi_2|_{(2,-1,2)} = \vec{n_2} = 4\hat{i} - 2\hat{j} - \hat{k}$

The vectors $\vec{n_1}$ and $\vec{n_2}$ are along normals to the surfaces at the point $(2,-1,2)$

If θ is the angle between these vectors then

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|} = \frac{(4\hat{i}-2\hat{j}+4\hat{k})(4\hat{i}-2\hat{j}-\hat{k})}{\sqrt{16+4+16}\cdot\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

$$\therefore \boxed{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

Q 2a Find the directional derivative of $\frac{1}{r}$ in the direction $\vec{r}$ where $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

Soln $\phi(x,y,z) = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$ (as $r = |\vec{r}|$)

$$\nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2+y^2+z^2)^{-1/2}$$

$$= \left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x\right)\hat{i} + \left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2y\right)\hat{j} + \left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2y\right)\hat{k}$$

$$= \frac{-(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}}$$

$\hat{r}$ = unit vector in the direction of $x\hat{i}+y\hat{j}+z\hat{k}$

$$= \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

so required directional derivative is

$$\boxed{\nabla\phi\cdot\hat{r}} = \frac{-(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{(x^2+y^2+z^2)^3}}\cdot\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \boxed{\frac{1}{r^2}}$$

Q 2b Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$. In what direction it will be maximum? Find also the magnitude of this maximum

Soln. we have $\nabla f = \hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z} = 2x\hat{i}-2y\hat{j}+4z\hat{k}$

$= 2\hat{i}-4\hat{j}+12\hat{k}$ at $P(1,2,3)$

Also $\vec{PQ} = \vec{OQ}-\vec{OP} = (5\hat{i}+4\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$

$= 4\hat{i}-2\hat{j}+\hat{k}$

If $\hat{n}$ is a unit vector in the direction $\vec{PQ}$, then

$$\hat{n} = \frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{16+4+1}} = \frac{1}{\sqrt{21}}(4\hat{i}-2\hat{j}+\hat{k})$$

$\therefore$ Directional derivative of f in the direction $\vec{PQ} = (\nabla f)\cdot\hat{n}$

$$= (2\hat{i}-4\hat{j}+12\hat{k})\cdot\frac{1}{\sqrt{21}}(4\hat{i}-2\hat{j}+\hat{k})$$

$$= \frac{1}{\sqrt{21}}[2(4)-4(-2)+12(1)] = \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}$$

The directional derivative of f is maximum in the direction of the normal to the given surface i.e, in the direction of $\nabla f = 2\hat{i}-4\hat{j}+12\hat{k}$

The maximum value of this directional derivative $= |\nabla f|$

$$= \sqrt{2^2+(-4)^2+12^2} = \sqrt{4+16+144}$$

$$= \sqrt{164}$$

$$= 2\sqrt{41}.$$

Q.3 Prove that $(y^2-z^2-3yz-2x)\hat{i}+(3xz+2xy)\hat{j}+(3xy-2xz+2z)\hat{k}$ is both solenoidal & irrotational.

Sol^n. Let $\vec{F}=(y^2-z^2-3yz-2x)\hat{i}+(3xz+2xy)\hat{j}+(3xy-2xz+2z)\hat{k}$

For solenoidal $\vec{\nabla}\cdot\vec{F}=0$

$$\vec{\nabla}\cdot\vec{F}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left((y^2-z^2-3yz-2x)\hat{i}+(3xz-2xy)\hat{j}+(3xy-2xz+2z)\hat{k}\right)$$

$$\vec{\nabla}\cdot\vec{F}=-2+2x-2x+2=0.$$

Thus $\vec{F}$ is solenoidal.

For irrotational $\text{curl}\,\vec{F}=\vec{\nabla}\times\vec{F}=0$

$$\Rightarrow \text{curl}\,\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (y^2-z^2+3yz-2x) & (3xz+2xy) & (3xy-2xz+2z)\end{vmatrix}$$

$$=\hat{i}(3x-3x)-\hat{j}(3y-2z+2z-3y)\hat{j}+(3z+2y-2y-3z)\hat{k}$$

$$=0\hat{i}+0\hat{j}+0\hat{k}=0$$

$\Rightarrow \vec{\nabla}\times\vec{F}=0$, Thus $\vec{F}$ is irrotational.

Hence both solenoidal & irrotational.

Q.4(a) Show that $\vec{A}=(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$ is irrotational. Find the velocity potential ϕ such that $\vec{A}=\nabla\phi$.

Given $\vec{A}=(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$

For irrotational $\text{curl}\,\vec{A}=\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 6xy+z^3 & 3x^2-z & 3xz^2-y\end{vmatrix}$

$$=\hat{i}(-1+1)+\hat{j}(3z^2-3z^2)+\hat{k}(6x-6x)$$

$$=0\hat{i}+0\hat{j}+0\hat{k}$$

$\Rightarrow \text{curl}\,\vec{A}=0$

$\Rightarrow A$ is irrotational

Since $\vec{A}$ is irrotational, so there exists a potential function ϕ such that $\vec{A}=\nabla\phi$

$$(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

$$\frac{\partial\phi}{\partial x}=6xy+z^3,\quad \frac{\partial\phi}{\partial y}=3x^2-z,\quad \frac{\partial\phi}{\partial z}=3xz^2-y$$

Now $d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$

$$d\phi=(6xy+z^3)dx+(3x^2-z)dy+(3xz^2-y)dz$$

Integrating

$$\int d\phi=\int(6xy+z^3)dx+\int(3x^2-z)dy+\int(3xz^2-y)dz$$

$$\phi=\int d(3x^2y)+d(xz^3)+d(-yz)+c$$

$$\phi=3x^2y+xz^3-yz+c$$

Q.4(b): Show that the velocity field $\vec{F}=\frac{\vec{r}}{r^3}$ is irrotational as well as solenoidal. Find the scalar potential.

Sol^n. Let $\vec{F}$ is to be irrotational $\text{curl}\,\vec{F}=0$

$$\text{curl}\left(\frac{1}{r^3}\cdot\vec{r}\right)=\frac{1}{r^3}\text{curl}\,\vec{r}+\text{grad}\left(\frac{1}{r^3}\right)\times\vec{r}$$

$(\because \text{curl}(u\vec{a})=u\,\text{curl}\,\vec{a}+\text{grad}\,u\times\vec{a})$

$$=\frac{1}{r^3}(\vec{0})+\left(\frac{-3}{r^4}\hat{r}\right)\times\vec{r}$$ $|\because \text{curl}\,\vec{r}$

$$=\vec{0}-\frac{3}{r^5}(\vec{r}\times\vec{r})$$

$$=\vec{0}\quad(\because \vec{r}\times\vec{r}=0)$$

Hence vector field $\vec{F}$ is irrotational.

For solenoidal $\operatorname{div} \vec{F} = 0$

$$\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\operatorname{div}\vec{r} + \vec{r}\,\operatorname{grad}\left(\frac{1}{r^3}\right)$$

$\because \operatorname{div}(u\vec{a}) = u \operatorname{div}\vec{a} + \vec{a}\operatorname{grad} u$

$$= \operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\operatorname{div}\vec{r} + \vec{r}\cdot\operatorname{grad}\left(\frac{1}{r^3}\right)$$

$$= \frac{3}{r^3} + \vec{r}\cdot\left(-\frac{3}{r^4}\cdot\frac{\vec{r}}{r}\right) \quad \Big|\ \because \operatorname{div}\vec{r} = 3$$

$$= \frac{3}{r^3} - \frac{3}{r^5}r^2$$

$$= \frac{3}{r^3} - \frac{3}{r^3} = 0$$

$\vec{F}$ is solenoidal

let $\vec{F} = \nabla\phi$ where ϕ is scalar potential

$$\vec{F}\cdot d\vec{r} = \nabla\phi\cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = d\phi$$

$$d\phi = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}} = d\left[-(x^2+y^2+z^2)^{-1/2}\right]$$

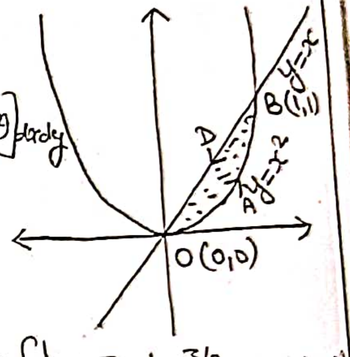
integrating we get $\boxed{\phi = -\frac{1}{\sqrt{x^2+y^2+z^2}} + C}$

$$\Rightarrow \boxed{\phi = -\frac{1}{r} + C}$$

* Best of Luck *

Q5 Verify the Green's Theorem to evaluate the line integral $\int_C (2y^2dx + 3x\,dy)$ where C is the boundary of the closed region bounded by $y = x$ & $y = x^2$

Ans Using Green's theorem



$$\int_C 2y^2dx + 3x\,dy = \iint\left[\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(2y^2)\right]dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=y}^{\sqrt{y}}(3-4y)\,dx\,dy$$

$$= \int_0^1 (3x - 4xy)_y^{\sqrt{y}}\,dy = \int_0^1 3\sqrt{y} - 4y^{3/2} - 3y + 4y^2\,dy$$

$$= \left[3\cdot\frac{2}{3}y^{3/2} - 4\cdot\frac{2}{5}y^{5/2} - \frac{3y^2}{2} + \frac{4y^3}{3}\right]_0^1 = \frac{7}{30}$$

Verification $\int_C 2y^2dx + 3x\,dy = \int_{OAB}(2y^2dx + 3x\,dy) + \int_{BDO}(2y^2dx + 3x\,dy)$

Along OAB $y = x^2 \Rightarrow dy = 2x\,dx$ and $x: 0 \to 1$

Along BDO $y = x \Rightarrow dy = dx$ and $x: 1 \to 0$

$$\int_C 2y^2dx + 3x\,dy = \int_{x=0}^{1}(2x^4 + 3x(2x))dx + \int_{x=1}^{0} 2x^2dx + 3x\,dx$$

$$= \int_0^1 (2x^4 + 6x^2)dx + \int_1^0 (2x^2 + 3x)dx$$

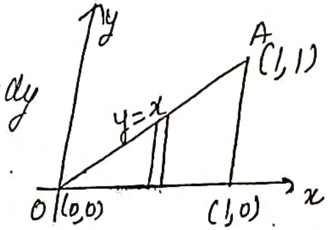
$$= \left(\frac{2x^5}{5} + 2x^3\right)_0^1 + \left(\frac{2}{3}x^3 + \frac{3}{2}x^2\right)_1^0$$

$$= \left(\frac{2}{5} + 2\right) + \left(-\frac{2}{3} - \frac{3}{2}\right) = \frac{12}{5} - \frac{13}{6} = \frac{7}{30}$$

Hence Green's theorem Verified.

Ques. 5: Using Green's Theorem, evaluate $\int_C (x^2y\,dx + x^2dy)$, where C is the boundary described counter clockwise of the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

Soln:- By Green's Theorem, we have



$$\int_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

$$\int_C (x^2y\,dx + x^2dy) = \iint_R (2x - x^2)\,dx\,dy$$

$$= \int_0^1 (2x-x^2)\,dx \int_0^x dy$$

$$= \int_0^1 (2x-x^2)\,dx\,[y]_0^x = \int_0^1 (2x-x^2)\,x\,dx$$

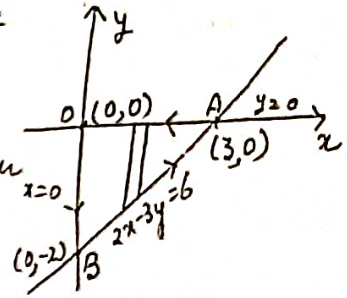
$$= \int_0^1 (2x^2-x^3)\,dx = \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^1 = \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{12} \text{ Ans.}$$

Ques 6:- State and verify Green's th. in the plane for $\oint (3x^2-8y^2)dx + (4y-6xy)dy$ where C is the boundary of the region bounded by $x \geq 0$, $y \leq 0$ and $2x-3y=6$.

Soln:- Statement:- If $\phi(x,y)$ and $\psi(x,y)$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ be continuous functn's over a region R bounded by simple closed curve C in x-y plane, then

$$\boxed{\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy}$$

Here the closed curve C consist of st. lines OB, BA and AO, where co-ordinates of A are $(3,0)$ and B are $(0,-2)$ respectively. Let R be the region bounded by C.



Then by Green's th. we have

$$\oint (3x^2-8y^2)dx + (4y-6xy)dy = \iint_R \left[\frac{\partial}{\partial x}(4y-6xy) - \frac{\partial}{\partial y}(3x^2-8y^2)\right] dx\,dy \quad (1)$$

$$= \iint_R (-6y+16y)\,dx\,dy = \iint_R 10y\,dx\,dy$$

$$= 10\int_0^3 dx \int_{\frac{1}{3}(2x-6)}^0 y\,dy = 10\int_0^3 dx \left[\frac{y^2}{2}\right]_{\frac{1}{3}(2x-6)}^0$$

$$= -\frac{5}{9}\int_0^3 (2x-6)^2 dx = -\frac{5}{9}\left[\frac{(2x-6)^3}{3\times 2}\right]_0^3$$

$$= -\frac{5}{54}(0+6)^3 = -\frac{5}{54}(216) = -20. \quad (2)$$

Verification: Now, we evaluate LHS of (1) along OB, BA and AO.

Along OB $x=0$, $dx=0$ and y varies from 0 to -2.

Along BA, $x = \frac{1}{2}(6+3y)$, $dx = \frac{3}{2}dy$ and y varies from -2 to 0.

Along AO, $y=0$, $dy=0$ and x varies from 3 to 0.

$$LHS = \oint [(3x^2-8y^2)dx + (4y-6xy)dy]$$

$$= \int_{OB} [(3x^2-8y^2)dx + (4y-6xy)dy] + \int_{BA} [(3x^2-8y^2)dx + (4x-6xy)dy]$$
$$+ \int_{AO} [(3x^2-8y^2)dx + (4y-6xy)dy]$$

$= \int_0^{-2} 4y\,dy + \int_{-2}^{0}\left[\frac{3}{4}(6+3y)^2 - 8y^2\right]\left(\frac{3}{2}dy\right) + [4y - 3(6+3y)y]dy + \int_3^0 3x^2 dx$

$= [2y^2]_0^{-2} + \int_{-2}^{0}\left[\frac{9}{8}(6+3y)^2 - 12y^2 + 4y - 18y - 9y^2\right]dy + [x^3]_3^0$

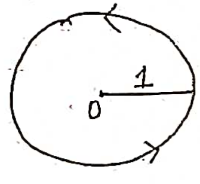
$= 8 + \left[\frac{9}{8}\frac{(6+3y)^3}{3\times 3} - 7y^3 - 7y^2\right]_{-2}^{0} - 27$

$= -19 + 27 - 56 + 28 = -20$

Thus Green's th. is verified.

Ques7:- Using Stoke's th. or otherwise, evaluate $\int_C [(2x-y)dx - yz^2dy - y^2z\,dz]$ where C is the circle $x^2+y^2=1$, corresponding to the surface of sphere of unit radius.

Soln:- $\int_C [(2x-y)dx - yz^2dy - y^2z\,dz] = \int_C [(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}]\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$



By Stoke's th. $\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds$ — (1)

$\text{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$

$= (-2yz + 2yz)\hat{i} - (0-0)\hat{j} + (0+1)\hat{k} = \hat{k}$

using (1) $= \iint \hat{k}\cdot\hat{n}\,ds = \iint \hat{k}\cdot\hat{n}\frac{dx\,dy}{(\hat{n}\cdot\hat{k})} = \iint dx\,dy$

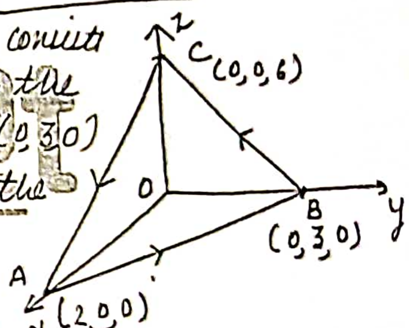
$= \iint$ Area of the circle $= \pi\cdot 1^2 = \underline{\pi \text{ Ans}}$ $\left[\because ds = \frac{dx\,dy}{(\hat{n}\cdot\hat{k})}\right]$

Ques18:- State and verify Stoke's th. $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$ for the surface of the triangular lamina with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$.

Soln:- Statement:- Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by the curve C is equal to the line integral of the vector point funct^n $\vec{F}$ taken along the closed curve C.

Mathematically, $\boxed{\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds}$.

$\Rightarrow$ Here the path of integrat^n c consists of the st. lines AB, BC, CA where the co-ordinates of A, B, C are $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$ respectively. Let S be the plane surface of triangle ABC bounded by C. Let $\hat{n}$ be the unit normal vector to surface S.



Then, by Stoke's th, we have

$\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds$ — (1)

LHS: $= \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r} + \int_{CA}\vec{F}\cdot d\vec{r}$

Along AB, $z=0 \Rightarrow \frac{x}{2} + \frac{y}{3} = 1 \Rightarrow y = \frac{3}{2}(2-x)$, $dy = -\frac{3}{2}dx$

At A, $x=2$, At B, $x=0$, $\vec{r} = x\hat{i} + y\hat{j}$

$\int_{AB}\vec{F}\cdot d\vec{r} = \int_{AB}(x+y)dx + 2x\,dy$

$$\int_{AB} \left(x+3-\frac{3x}{2}\right)dx+(2x)\left(-\frac{3}{2}dx\right) = \int_2^0 \left(-\frac{7x}{2}+3\right)dx = 1.$$

Along BC, $x=0$, $\frac{y}{3}+\frac{z}{6}=1$ or $z=6-2y$, $dz=-2dy$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_{BC} -z\,dy+(y+z)\,dz = \int_3^0 (-6+2y)dy+(y+6-2y)(-2dy)$$
$$= \int_3^0 (4y-18)dy = 36.$$

Along CA, $y=0$, $\frac{x}{2}+\frac{z}{6}=1$ or $z=6-3x$, $dz=-3dx$

$$\int_{CA} \vec{F}\cdot d\vec{r} = \int_{CA} (x\,dx+z\,dz) = \int_0^2 x\,dx+(6-3x)(-3)dx$$
$$= \int_0^2 (10x-18)dx = -16$$

$\therefore$ LHS of ① $= \int_{ABC} \vec{F}\cdot d\vec{r} = 1+36-16 = 21$

Also, $\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\hat{i}+\hat{k}$

Eqn of the plane ABC is $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$

Normal to the plane ABC is $\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{6}-1\right)$

$$= \frac{\hat{i}}{2}+\frac{\hat{j}}{3}+\frac{\hat{k}}{6}$$

Unit normal vector $= \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}} = \hat{n}$ (say)

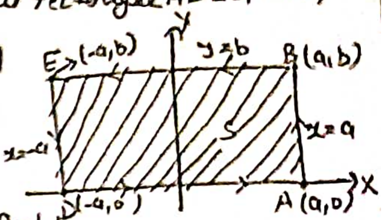
RHS of ① $= \iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,ds = \iint_S (2\hat{i}+\hat{k})\cdot\frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})\frac{dx\,dy}{\left(\frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})\cdot\hat{k}\right)}$

$$= \iint 7\,dx\,dy = 7\cdot\text{Area of }\Delta OAB$$
$$= 7\left(\frac{1}{2}\cdot 2\times 3\right) = 21 \text{ Ans}$$

Thus, the Stoke's th. is verified.

Q.(b) Verify Stoke's theorem for $\vec{F} = (x^2+y^2)\hat{i}-2xy\hat{j}$ taken round the rectangle bounded by the lines $x=\pm a$, $y=0$, $y=b$

Sol^n: Let C denote the boundary of the rectangle ABED, then,



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C [(x^2+y^2)\hat{i}-2xy\hat{j}][\hat{i}dx+\hat{j}dy]$$
$$= \oint_C [(x^2+y^2)dx-2xy\,dy]$$

The curve C consists of four lines AB, BE, ED & DA.

Along AB, $x=a$, $dx=0$ & y varies from 0 to b.

$\therefore \int_{AB} [(x^2+y^2)dx-2xy\,dy] = \int_0^b -2ay\,dy = -a[y^2]_0^b = -ab^2$ — ①

Along BE, $y=b$, $dy=0$ & x varies from a to $-a$.

$\therefore \int_{BE} [(x^2+y^2)dx-2xy\,dy] = \int_a^{-a} (x^2+b^2)dx = \left[\frac{x^3}{3}+b^2x\right]_a^{-a} = -\frac{2a^3}{3}-2ab^2$ — ②

Along ED, $x=-a$, $dx=0$ & y varies from b to 0.

$\therefore \int_{ED} [(x^2+y^2)dx-2xy\,dy] = \int_b^0 2ay\,dy = a[y^2]_b^0 = -ab^2$ — ③

Along DA, $y=0$, $dy=0$ & x varies from $-a$ to a.

$\therefore \int_{DA} [(x^2+y^2)dx-2xy\,dy] = \int_{-a}^{a} x^2dx = \frac{2a^3}{3}$ — ④

Adding ①, ②, ③ & ④, we get

$$\oint_C \vec{F}\cdot d\vec{r} = -ab^2-\frac{2a^3}{3}-2ab^2-ab^2+\frac{2a^3}{3} = -4ab^2 \text{ — ⑤}$$

Now $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = (-2y-2y)\hat{k} = -4y\hat{k}$

For the surface S, $\hat{n}=\hat{k}$

$$\text{curl}\,\vec{F}\cdot\hat{n} = -4y\hat{k}\cdot\hat{k} = -4y$$

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \int_0^b\int_{-a}^{a} -4y\,dx\,dy = \int_0^b -4y[x]_{-a}^{a}dy$$
$$= -8a\int_0^b y\,dy = -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \text{ — ⑥}$$

The equality of ⑤ & ⑥ verifies Stoke's theorem.

Ques:- State Gauss's Divergence th. and find $\iint \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$ and S is the surface of the sphere having centre $(3,-1,2)$ and radius 3.

Soln:- Statement:- The surface integral of the normal component of a vector funct^n F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.

Mathematically, $\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV}$

$\Rightarrow$ Let V be the volume enclosed by the surface S.

By divergence th.; we have

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint \operatorname{div}\vec{F}\,dV \quad (1)$$

Now $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(2x+3z) + \frac{\partial}{\partial y}[-(xz+y)] + \frac{\partial}{\partial z}(y^2+2z)$

$= 2 - 1 + 2 = 3$

$\therefore$ from eq (1) $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V 3\,dV = 3V$

Again V is the volume of the sphere of radius 3.

$\therefore\ V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3)^3 = 36\pi$

Thus $\iint_S \vec{F}\cdot\hat{n}\,ds = 3V = 3\times 36\pi = 108\pi$ Ans.

Q-10 Verify divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$.

Soln:- Given $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ — (1)

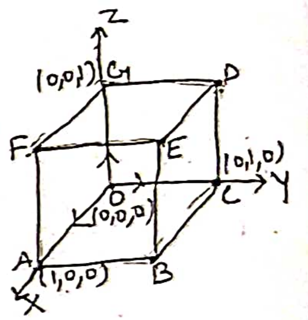
$\therefore \operatorname{div}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(4xz\hat{i} - y^2\hat{j} + yz\hat{k})$

$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y = 4z - y$

$\iiint_V \operatorname{div}\vec{F}\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z-y)\,dz\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1}[2z^2 - yz]_0^1\,dy\,dx$

$= \int_{x=0}^{1}\int_{y=0}^{1}(2-y)\,dy\,dx = \int_{x=0}^{1}\left[2y - \frac{y^2}{2}\right]_0^1 dx = \int_{x=0}^{1}\frac{3}{2}dx = \frac{3}{2}$ — (2)

To evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$.



Here six the surface of the cube bounded by the 6 plane surfaces.

over the face OABC, $z=0, dz=0, \hat{n} = -\hat{k}, ds = dx\,dy$

$\therefore \iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1(-y^2\hat{j})\cdot(-\hat{k})\,dx\,dy = 0$ — (3)

over the face BCDE, $y=1, dy=0, \hat{n} = \hat{j}, ds = dx\,dz$

$\therefore \iint_D \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1(4xz\hat{i} - \hat{j} + z\hat{k})\cdot\hat{j}\,dx\,dz$

$= -\int_0^1\int_0^1 dx\,dz - (x)_0^1(z)_0^1 = -1$ — (4)

over the face DEFG, $z=1, dz=0, \hat{n} = \hat{k}, ds = dx\,dy$

$\therefore \iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1(4x\hat{i} - y^2\hat{j} + y\hat{k})\cdot\hat{k}\,dx\,dy$

$= \int_0^1\int_0^1 y\,dx\,dy = \int_0^1 dx\int_0^1 y\,dy = \frac{1}{2}$ — (5)

over the face AOGF, $y=0, dy=0, \hat{n} = -\hat{j}, ds = dx\,dz$

$\therefore \iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1(4xz\hat{i})\cdot(-\hat{j})\,dx\,dz = 0$ — (6)

over the face OCDG, $x = 0$, $dx = 0$, $\hat{n} = -\hat{i}$, $dS = dy\,dz$

$$\therefore \iint \vec{F}\cdot\hat{n}\,dS = \int_0^1\int_0^1 (-y^2\hat{j} + yz\hat{k})\cdot(-\hat{i})\,dy\,dz = 0 \quad \text{--- (7)}$$

over the face ABEF, $x = 1$, $dx = 0$, $\hat{n} = \hat{i}$, $dS = dy\,dz$

$$\therefore \iint \vec{F}\cdot\hat{n}\,dS = \int_0^1\int_0^1 (4z\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{i}\,dy\,dz$$

$$= \int_0^1\int_0^1 4z\,dy\,dz = \int_0^1 dy\int_0^1 4z\,dz = 2 \quad \text{--- (8)}$$

Adding eqns (3), (4), (5), (6), (7) & (8), we get over the whole surface S,

$$\iint_S \vec{F}\cdot\hat{n}\,dS = 0 - 1 + \frac{1}{2} + 0 + 0 + 2 = \frac{3}{2} \quad \text{--- (9)}$$

from eqn (2) & (9), $\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV$

Hence the divergence theorem is verified.